Modelization and valuation methods of gas contracts: Stochastic control approaches

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Résumé

Le travail présenté dans cette thèse a été motivé par des problématiques posées par l'évaluation de contrats échangés sur le marché du gaz: les contrats de stockage et d'approvisionnement en gaz. Ceux-ci incorporent de l'optionalité et des contraintes, ce qui rend leur évaluation difficile dans un contexte de prix de matières premières aléatoires. L'évaluation de ces contrats mène à des problèmes de contrôle stochastique complexes: switching optimal ou contrôle impulsionnel et contrôle stochastique en grande dimension.

La première partie de cette thèse est une revue relativement exhaustive de la littérature, mettant en perspective les différentes approches d'évaluation existantes. Dans une deuxième partie, nous considérons une méthode numérique de résolution de problèmes de contrôle impulsionnel basée sur leur représentation comme solution d'EDSRs à sauts contraints. Nous proposons une approximation à temps discret utilisant une pénalisation pour traiter la contrainte et donnons un taux de convergence de l'erreur introduite. Combinée avec des techniques Monte Carlo, cette méthode a été testée numériquement sur trois problèmes: gestion optimale de biomasse, évaluation d'options Swing et de contrats de stockage gaz. Dans une troisième partie, nous proposons une méthode pour l'évaluation d'options dont le payoff dépend de moyennes mobiles de prix sous-jacents. Elle utilise sur une approximation à dimension finie de la dynamique des processus de moyenne mobile, basée sur un développement en série de Laguerre tronquée. Les résultats numériques fournis incluent des exemples de contrats Swing gaziers à prix d'exercice indexés sur moyennes mobiles de prix pétroliers.

Mots-Clés: Contrôle stochastique, Equations Différentielles Stochastiques Rétrogrades, Contrôle impulsionnel, Méthodes numériques, Evaluation d'actifs gaziers, Options réelles, Monte Carlo, Moyenne mobile.

Abstract

The work presented in this PhD dissertation was motivated by issues raised by the valuation of contracts commonly traded in the gas market: gas storage and gas supplying contracts. Both of them include optionality and constraints, which make tough their valuation in a context of uncertain commodity prices. The valuation of such contracts leads to particularly complex stochastic control problems: optimal switching or impulse control and optimal control in high dimension.

The first part of this dissertation is a relatively exhaustive review of the existing valuation methods. In a second part, we introduce a numerical method for solving impulse control problems by using their representations as BSDEs with constrained jumps. We propose a discrete-time approximation using a penalization of the constraint on the jump component and provide a rate of convergence of the approximation error. Combining this approach with Monte Carlo techniques, we perform numerical tests on three problems: optimal forest management, valuation of Swing options and gas storage facilities. In a third part, we introduce a method for pricing options whose payoff depends on moving averages of underlying prices. We use a finite-dimensional approximation of the infinite-dimensional dynamics of moving average processes, based on a truncated Laguerre series expansion. We present numerical results including examples of gas Swing contracts involving strike prices indexed on moving averages of oil prices.

Keywords: Stochastic control, Backward Stochastic Differential Equations, Impulse control, Numerical methods, Gas assets valuation, Real options, Monte Carlo, Moving average.

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Introduction générale

Le travail présenté dans cette thèse porte sur des méthodes d'évaluation de contrats échanges sur le marché du gaz. Nous nous sommes intéressés plus spécifiquement à deux types de contrats gaziers, qui sont parmi les plus échangés: les contrats de stockage et les contrats d'approvisionnement en gaz. Ceux-ci incorporent de l'optionalité et des contraintes, ce qui rend leur évaluation difficile dans un contexte de prix de matières premières aléatoires. Ce sont des questions qui se posent dans le contexte européen actuel d'un marché dérégulé du gaz et qui ont été motivées par des problématiques pratiques rencontrés par Électricité de France[1], fournisseur historique d'électricité en France qui a diversifié ses activités à la fourniture en gaz depuis une dizaine d'années.

D'un point de vue mathématique, l'évaluation de ce type de contrats (options réelles) mène à des problèmes de contrôle stochastique particulièrement complexes. Il s'agit de problèmes d'arrêt optimal multiple, de contrôle stochastique séquentiel (problèmes dits de *switching* optimal) ou plus généralement de problèmes de contrôle impulsionnel. Par ailleurs, les spécificités des contrats gaziers étudiés ici font intervenir des difficultés supplémentaires, à la fois d'un point de vue numérique et théorique.

D'une part, nous avons identifié des contraintes non triviales intervenant dans les contrats de stockage qui sont dus aux caractéristiques opérationnelles des actifs de stockage. L'évaluation d'actifs de stockage de gaz constitue mathématiquement un problème de *switching* optimal à trois régimes qui fait intervenir une variable dégénérée, contrôlée et contrainte (le niveau de stock dans l'actif). Les méthodes utilisés classiquement nécessitent des temps de calcul relativement longs: un défi majeur est donc de proposer de nouvelles méthodes d'évaluation moins chère en temps de calcul. D'autre part, les contrats d'approvisionnement en gaz sont plus communément connus sous le nom d'options Swing indexées: le payoff de ces contrats dépend, par le biais du prix d'exercice, de moyennes mobiles de prix de matières premières (gasoil, fuel, etc.). Il en résulte un problème d'arrêt optimal multiple en dimension infinie (à temps continu). A temps discret, le défi est encore une fois numérique puisque se pose le problème de la (grande) dimension.

Nous sommes partis de ces deux problématiques différentes pour arriver à deux problèmes théoriques qui peuvent être chacun posés dans un cadre mathématique plus général. En premier lieu, nous considérons une méthode numérique pour la résolution de problèmes de contrôle impulsionnel basée sur leur représentation comme solution d'une certaine classe d'Equations Différentielles Stochastiques Rétrogrades (EDSRs): il s'agit d'EDSRs à sauts contraints. En second lieu, nous proposons une approche pour l'évaluation d'options sur moyenne mobile basée sur une approximation à dimension finie de la dynamique à dimension infinie des processus de moyenne mobile.

Dans cette thèse, nous introduisons donc principalement deux nouvelles méthodes de résolution et dans les deux cas, notre approximation est justifiée théoriquement en incluant des vitesses de convergence explicites de la solution approchée vers la solution exacte.

Dans la suite de cette introduction, on présente les deux problématiques distinctes mentionnées ci-dessus. Après une description succincte des contrats gaziers considérés, les difficultés principales que nous avons identifiées sont soulignées et nous présentons nos contributions: résultats à la fois théoriques et numériques.

[1]Cette thèse a été effectuée dans le cadre d'un partenariat entre EDF R&D et le Laboratoire de Probabilités et Modèles Aléatoires CNRS-Universités Paris 6-Paris 7.

0.1 Une approche par EDSRs pour la résolution de problèmes de contrôle impulsionnel

0.1.1 Le cas particulier des contrats de stockage de gaz

Dans le cadre des options réelles, on peut voir la valeur de non-arbitrage d'un actif de stockage de gaz comme le profit maximal que peut espérer le détenteur du stockage s'il opère l'actif de façon optimale: l'évaluation de stockage est modélisée du point de vue de l'opérateur du stockage qui doit gérer l'actif de façon à maximiser ses bénéfices.

En chaque date précédant l'expiration du contrat, le détenteur du stockage observe le prix du gaz sur le marché spot et détermine la stratégie optimale parmi trois actions possibles: injecter, soutire ou ne rien faire. On dit que le stockage possède trois régimes (ou modes) d'opération. La cavité est relié aux points de livraison par le biais de pipelines. Si la stratégie est d'injecter, du gaz est acheté sur le marché et injecté dans la cavité. S'il s'agit de soutirer, du gaz est pompé du stockage et revendu sur le marché. Toutes ces opérations sont soumises à des contraintes de volumes, des contraintes physiques et opérationnelles telles que des capacités maximales d'injection et de soutirage et des coûts divers (en particulier, quand l'actif passe d'un mode d'opération à un autre).

Les bénéfices induits par la détention d'un actif de stockage sont linéaires en les capacités (ou taux) d'injection et de soutirage. Par conséquent, on peut démontrer que la stratégie optimale est de type "bang-bang" (à temps continu). En d'autres termes, il est toujours optimal d'injecter ou de soutirer aux taux maximaux possibles. Le problème d'évaluation d'un actif de stockage se ramène donc, à temps continu, à un problème de *switching optimal à 3 régimes*.

La commande de ce problème de contrôle stochastique correspond à la stratégie de *switching* et la variable d'état est composé du prix spot du gaz (ou des différents facteurs explicatifs, dans le cas d'un modèle de prix multi-facteurs) et du niveau de stock. Cette dernière variable a trois caractéristiques qui rendent le problème de *switching* plus complexe autant d'un point théorique que numérique: elle est *contrôlée* par la stratégie de *switching* (le niveau de stock de gaz croît en injection et décroît en soutirage), *contrainte* (à causes des contraintes de volumes imposées dans la cavité) et *dégénérée* en ce sens que son coefficient de diffusion est nul.

Pour surmonter la difficulté introduite par la variable de stock, on peut avoir recours à une discrétisation de l'ensemble des niveaux de stock admissibles. Cette approche classique est utilisée entre autres par Barrera-Esteve et al. [2] et Warin [27], mais est particulièrement coûteuse en place mémoire car il est nécessaire de calculer et garder en mémoire la solution en chaque point de cette grille discrète. Makassikis et al. [23] réussissent à fortement accélérer un tel type de méthode d'évaluation de stockage en utilisant un algorithme de parallélisation. Par ailleurs, une telle résolution sur une grille discrète à la fois *fixe et globale* engendre une perte d'efficacité numérique, puisque l'exploration de certaines régions du stock est inutile pour déterminer la solution optimale (les régions qui ne sont pas traversés par le niveau de stock optimal).

Une autre approche est possible: Carmona et Ludkovski [8] proposent une méthode de moindres carrés Monte Carlo bivarié, selon laquelle les espérances conditionnelles qui interviennent dans l'algorithme de résolution rétrograde sont calculées par une régression bivariée sur le prix de gaz courant et le niveau de stock. Une telle approche nécessite la simulation de trajectoires de niveaux de stock, en d'autres termes de deviner le niveau de stock à chaque date de la résolution rétrograde: dans [8], une approche combinant randomisation et conjecture (i.e. pré-supposition)

de la stratégie optimale est utilisée. Il s'agit d'une approximation basée sur une heuristique et dont la justification théorique semble difficile. Dans la même direction de recherche que Carmona et Ludkovski [8], notre volonté est de proposer une approche de résolution permettant d'éviter la discrétisation en niveau de stock, c'est-à-dire une méthode alternative de type Monte Carlo pure utilisant à la fois des trajectoires de prix du gaz et de niveaux de stock.

Il nous faut mentionner ici que ce type de problème de *switching* optimal est un cas particulier du *contrôle impulsionnel* en plus grande dimension, spécifiquement si l'on ajoute au système un processus purement discontinu, contrôlé par la stratégie de *switching* (variable à sauts purs) représentant en toute date le régime d'opération courant. Par conséquent, ces considérations soulèvent une question plus générale: il s'agit de trouver des méthodes probabilistes alternatives pour la résolution de problèmes de contrôle impulsionnel dans lesquels la variable d'état est contrôlée par la stratégie impulsionnelle et potentiellement générée. Les récents développements de la théorie des EDSRs et leurs applications possibles à de tels types de problèmes ouvrent de nouvelles perspectives qui vont dans ce sens. Notamment, d'un point de vue numérique, les techniques de résolution de type Monte Carlo pure sont la façon naturelle de résoudre les EDSRs.

0.1.2 Résolution de problèmes de contrôle impulsionnel basée sur les EDSRs

Les EDSRs fournissent des caractérisations alternatives des solutions des problèmes de *switching* optimal et plus généralement de contrôle impulsionnel. Les EDSRs reliées à ce type de problèmes sont particulièrement complexes et ce domaine de recherche reste relativement inexploré d'un point de vue numérique, à cause des difficultés rencontrées en pratique dans la résolution de telles EDSRs.

Dans le cas particulier du *switching* optimal, ce sont des EDSRs multidimensionelles réfléchies (réflexion de type oblique), voir par exemple Hu et Tang [18] et Hamadène et Zhang [17]. Chassagneux et al. [11] ont récemment introduit un schéma numérique rétrograde de résolution de telles EDSRs en ayant recours au concept d'EDSRs à réflexions obliques discrètes. Ces derniers obtiennent une vitesse de convergence de l'approximation en $|\pi|^{\frac{1}{2}-\epsilon}$ pour tout $\epsilon > 0$, quand le pas de la grille de discrétisation vaut $|\pi|$. Cependant, ce résultat est démontré dans un cadre où la variable d'état du système stochastique est *non contrôlée*.

Dans le cadre plus général du contrôle impulsionnel, les EDSRs associées sont des EDSRs à sauts contraints, qui ont été introduits par Kharroubi et al. [20]. Cette représentation reste valable pour le *switching* optimal comme le soulignent Elie et Kharroubi [13]. De plus, une telle caractérisation permet une dynamique contrôlée et dégénérée de la variable d'état du problème de contrôle impulsionnel. La principale difficulté dans l'approximation numérique de telles EDSRs provient de la contrainte qui concerne la composante de sauts de leur solution. En particulier, ces EDSRs ne font intervenir a priori aucune condition de minimalité de type Skorohod. Les approches classiques par des schémas par projection (schémas rétrogrades discrètement réfléchis) ne sont donc plus pertinents.

Une alternative consiste à utiliser une *procédure de pénalisation*: c'est-à-dire que la contrainte sur les sauts est introduite dans le *driver* de l'EDSR et pénalisée par un paramètre $p > 0$: quand la contrainte est satisfaite ce terme supplémentaire de pénalisation disparaît, sinon il pénalise le *driver* par un facteur explosif. La solution de l'EDSR à sauts pénalisée associée converge vers la solution minimale de l'EDSR à sauts contraints, voir [20] et [13]. Cependant, aucune vitesse de convergence n'existe pour une telle approximation. Cette approche par pénalisation peut

également être vue comme une alternative aux schémas itératifs classiquement utilisés basés sur une itération sur le nombre d'interventions, considérés par exemple par Carmona et Touzi [9] dans le cas des options Swing ou par Chancelier et al. [10] et Seydel [26] dans un cadre général de contrôle impulsionnel (approches de résolution non probabilistes).

D'un point de vue pratique, peu de tests numériques faisant intervenir des EDSRs pour la résolution de ce type de problèmes ont été publiés dans la littérature [25, 16, 22]. Dans toutes ces références, les variables d'état sont encore une fois non contrôlées.

Porchet [25] utilise un schéma par projection pour résoudre l'EDSR bidimensionnelle réfléchie associée à un problème d'option réelle (évaluation d'une centrale électrique à deux régimes). Le cas du *switching* optimal à 2 régimes simplifie fortement le calcul de la solution car l'EDSR bidimensionnelle réfléchie associée (la réflexion devient de type normale) peut être reformulée comme une seule EDSR réfléchie à deux barrières (en considérant le processus valeur correspondant à la différence entre les deux processus valeurs solutions). Hamadène et Jeanblanc [16] s'intéressent également à un problème de type *starting-stopping* et ont recours à une procédure de pénalisation pour résoudre l'EDSR réfléchie à deux barrières. Dans un cadre spécifique (en particulier: variable d'état non contrôlée et coûts de switching constants), il est démontré que la vitesse de convergence de l'erreur entre la solution de l'EDSR pénalisée et la solution exacte est en p^{-1}.

Enfin, Ludkovski [22] résout un problème de *switching* optimal à 3 régimes en considérant une cascade d'EDSRs réfléchies à une barrière en utilisant une itération sur le nombre de changements de régime, voir également Carmona et Ludkovski [7]. En se basant sur leurs observations numériques, les auteurs intuitent que l'erreur globale de leur méthode numérique croît linéairement avec le nombre maximal de changements de régime.

D'autre part, un schéma rétrograde à temps discret a été introduit par Bouchard et Elie [4] pour l'approximation numérique d'EDSRs à sauts (sans contrainte). Sous des hypothèses standards sur les coefficients de l'EDSR et de la diffusion à sauts sous-jacente, ils obtiennent une vitesse de convergence en $|\pi|^{\frac{1}{2}}$. De plus, Elie [12] présente des tests numériques utilisant les EDSRs à sauts: l'auteur fait notamment allusion au rôle critique de l'intensité λ de la mesure de sauts.

Suite à ces observations, nous avions plusieurs objectifs. Tout d'abord, nous souhaitions fournir une vitesse de convergence de l'erreur introduite par la procédure de pénalisation décrite précédemment. Cela nous permettrait dans un second temps d'obtenir une vitesse de convergence globale de l'erreur entre la solution du problème de contrôle impulsionnel considéré et l'approximation donnée par la solution de l'EDSR à sauts pénalisée résolue selon un schéma rétrograde à temps discret. Notre objectif était également de fournir une telle vitesse de convergence explicite en fonction des divers paramètres d'approximation introduits: c'est-à-dire, l'intensité des sauts λ, le coefficient de pénalisation p et le pas de temps $|\pi|$. Une telle estimation d'erreur est par ailleurs essentielle en pratique (meilleure compréhension de l'impact numérique de ces paramètres) et permet typiquement d'ajuster la maille de la grille de discrétisation en temps en fonction de (λ, p). Enfin, nous voulions éprouver l'efficacité de cette méthode numérique sur des problèmes d'options réelles.

0.1.3 Une approche par pénalisation pour la résolution d'EDSRs à sauts contraints

Nous considérons le problème de contrôle impulsionnel suivant:

$$v(t,x) = \sup_{u=(\tau_k)_{k\geq 1}\in\mathcal{U}_{(t,T]}} \mathbb{E}\left[g(X_T^{t,x,u}) + \int_t^T f(X_s^{t,x,u})ds + \sum_{\substack{k\geq 1\\ t<\tau_k\leq T}} \kappa(X_{\tau_k^-}^{t,x,u})\right]. \tag{1}$$

Un contrôle $u = (\tau_k)_{k\geq 1}$ est une suite croissante de temps d'arrêt correspondants aux dates d'intervention et la variable d'état contrôlé X^u est un processus càdlàg tel que:

$$X_t^u = X_0^u + \int_0^t b(X_s^u)ds + \int_0^t \sigma(X_s^u)dW_s + \sum_{\tau_k\leq t} \gamma(X_{\tau_k^-}^u), \quad \forall t \geq 0, \tag{2}$$

où W est une mouvement Brownien d-dimensionnel. Sous des hypothèses pertinentes, Kharroubi et al. [20] montrent que la solution du problème (1) peut être représentée comme la solution minimale de l'EDSR à sauts contraints:

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s - \int_t^T V_s dN_s + \int_t^T dK_s, \ \forall 0 \leq t \leq T \\ -V_t \geq \kappa(X_{t^-}), \ \forall 0 \leq t \leq T \end{cases} \tag{3}$$

dans laquelle N est un processus de Poisson d'intensité $\lambda > 0$ et X est une diffusion à sauts suivant la dynamique:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \gamma(X_{t^-})dN_t. \tag{4}$$

La solution (Y, Z, V, K) est dite *minimale* si c'est celle qui a la plus petite composante Y dans la classe (infinie) des solutions de (3). Mentionnons à cette occasion que cette caractéristique de la solution ne semble pas conciliable avec une résolution numérique.

Nous introduisons donc une approche de résolution de l'EDSR à sauts contraints (3), basée sur la pénalisation de la contrainte d'obstacle. Plus précisément, on se donne un paramètre de pénalisation $p > 0$ et on se ramène à une EDSR à sauts plus simple:

$$Y_t^p = g(X_T) + \int_t^T \left[f(X_s) + p\left(V_s^p + \kappa(X_{s^-})\right)^+ \lambda\right] ds - \int_t^T Z_s^p dW_s - \int_t^T V_s^p dN_s, \tag{5}$$

dont l'unique solution (Y^p, Z^p, V^p) tend vers la solution minimale (Y, Z, V) de (3) quand $p \to +\infty$. Etant donné une grille de discrétisation en temps régulière $\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}$, nous considérons alors le schéma de résolution rétrograde suivant pour la résolution numérique de l'EDSR à sauts pénalisée (3), voir Bouchard et Elie [4]:

$$\begin{cases} \bar{Y}_{t_N}^{p,\pi} = g(X_{t_N}^\pi) \\ \forall t_n \in \pi, t_n < T: \\ \qquad \bar{V}_{t_n}^{p,\pi} = \frac{1}{\lambda\Delta t_{n+1}}\mathbb{E}\left[\bar{Y}_{t_{n+1}}^{p,\pi}\Delta\tilde{N}_{t_{n+1}}|\mathcal{F}_{t_n}\right] \\ \qquad \bar{Z}_{t_n}^{p,\pi} = \frac{1}{\Delta t_{n+1}}\mathbb{E}\left[\bar{Y}_{t_{n+1}}^{p,\pi}\Delta W_{t_{n+1}}|\mathcal{F}_{t_n}\right] \\ \qquad \bar{Y}_{t_n}^{p,\pi} = \mathbb{E}\left[\bar{Y}_{t_{n+1}}^{p,\pi}|\mathcal{F}_{t_n}\right] + \left[f(X_{t_n}^\pi) + \left(p\left(\bar{V}_{t_n}^{p,\pi} + \kappa(X_{t_n}^\pi)\right)^+ - \bar{V}_{t_n}^{p,\pi}\right)\lambda\right]\Delta t_{n+1} \end{cases} \tag{6}$$

où $\Delta t_{n+1} = t_{n+1} - t_n$, $\Delta W_{t_{n+1}}$ est l'incrément Brownien sur $[t_n, t_{n+1}]$ et $\Delta \tilde{N}_{t_{n+1}}$ la version compensée de l'incrément de Poisson $\Delta N_{t_{n+1}}$.

Nous fournissons dans cette thèse un vitesse de convergence globale de l'erreur introduite par l'approximation numérique décrite ci-dessus, en fonction de λ, p et du pas de temps $|\pi| := \Delta t_{n+1}, \forall n < N$. Sous des hypothèses relativement générales sur les coefficients de l'EDSR et de la diffusion à sauts (Lipschitz continuité de g, f, κ, b, σ et γ et γ uniformément borné), nos principaux résultats sont les suivants:

- Dès que le nombre (aléatoire) d'interventions optimales du problème de contrôle impulsionnel considéré est dans L^2, nous montrons que la fonction valeur (1) est une fonction $\frac{1}{2}$-Hölder de la maturité. Ce résultat reste vrai dans le cadre plus général des problèmes de contrôle impulsionnel considérés dans Kharroubi et al. [20].

- Nous obtenons une vitesse de convergence de l'erreur dûe à la pénalisation sous des hypothèses supplémentaires appropriées. Soit $(Y^{p,t,x}, Z^{p,t,x}, V^{p,t,x})$, la solution de (5) quand $X \equiv (X_s^{t,x})_{t \le s \le T}$ est la solution partant de x à l'instant t de l'EDS (4). Nous utilisons une représentation explicite fonctionnelle pour $Y^{p,t,x}$ comme un essentiel supremum sur une famille de probabilités qui impacte seulement l'intensité de N. En ayant recours à un changement de probabilité bien choisi, qui force la solution pénalisée à sauter aussitôt que possible après qu'une impulsion optimale se produit, nous démontrons, en utilisant également la régularité de la fonction valeur en sa variable maturité, que:

$$\mathcal{E}^p = \sup_{0 \le t \le T} \left| v(t,x) - Y_t^{p,t,x} \right| \le C \left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} \right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right).$$

- Nous avons ensuite procédé à l'estimation de l'erreur de discrétisation introduite par le schéma numérique (6) en termes des paramètres supplémentaires λ et p, en utilisant les même arguments que Bouchard et Elie [4]. Sous des hypothèses standards, nous mettons en évidence le fait qu'une condition nécessaire pour une convergence en $|\pi|^{\frac{1}{2}}$ du schéma numérique à temps discret est

$$|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2} \right). \tag{7}$$

  Grâce à des arguments classiques de régularisation et de différentiation basée sur le calcul de Malliavin appliqué à l'EDSR à sauts pénalisée, cela nous permet de fournir une estimation explicite de l'erreur de discrétisation: celle-ci croît exponentiellement en (λp^2).

- Finalement, nous en déduisons une vitesse de convergence globale de l'approximation numérique introduite par notre procédure de pénalisation. En particulier,

$$\begin{aligned} \mathcal{E}^p + \mathcal{E}^\pi(Y^p) &= \sup_{0 \le t \le T} \left| v(t,x) - Y_t^{p,t,x} \right| + \left(\max_{n < N-1} \sup_{t_n \le t \le t_{n+1}} \mathbb{E} \left| Y_t^p - \bar{Y}_{t_n}^{p,\pi} \right|^2 \right)^{\frac{1}{2}} \\ &= \mathcal{O}_{p \to +\infty} \left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} + (1+\lambda)^2 \lambda p \bar{C}^{\lambda p^2} |\pi|^{\frac{1}{2}} \right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right) \end{aligned} \tag{8}$$

pour une certaine constante $\bar{C}$ supérieure à 1 qui ne dépend ni de λ, p, $|\pi|$ et α. Pour un pas de temps suffisamment petit $|\pi|$ relativement à λ et p, l'erreur globale est donc en

$$[\mathcal{E}^p + \mathcal{E}^\pi(Y^p)]^* = \mathcal{O}_{p\to+\infty}\left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}}\right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right). \tag{9}$$

Notre seconde contribution concerne l'utilisation en pratique d'une telle méthode. En combinant la méthode présentée ci-dessus avec une approche de type Monte Carlo, nous avons procédé à des tests numériques sur des cas pratiques de contrôle impulsionnel dans le cadre des options réelles. Nous fournissons par ailleurs des algorithmes entièrement implémentables. Les résultats que nous obtenons permettent une meilleure compréhension de l'impact de l'intensité des sauts et du coefficient de pénalisation d'un point de vue numérique:

- Nous considérons en premier lieu un problème de gestion optimale de biomasse forestière. Nous retrouvons numériquement le comportement de l'erreur d'approximation comme fonction de l'intensité λ à p fixé, cf. (9). Des résultats précis sont obtenus (une solution quasi-analytique est disponible pour ce problème) et la méthode est efficace sur cet exemple.

- Nous traitons ensuite le problème d'évaluation d'options à exercices anticipés multiples (options Swing normalisées). Ce problème d'arrêt optimal multiple engendre un problème de contrôle impulsionnel en dimension 3 particulièrement dégénéré. La méthode est clairement moins performante mais nous avons pu obtenir (quand le nombre de droits d'exercices est petit) des résultats d'évaluation stables et précis, en augmentant suffisamment le nombre de pas de temps et en utilisant assez de tirages Monte Carlo.

- La méthode numérique s'avère bien plus sensible au paramètre p qu'à λ, ce que l'on retrouve dans (7) et (8). En pratique, le paramètre de pénalisation doit être choisi relativement petit (inférieur à 5 dans les exemples que nous avons pris) et le pas de temps $|\pi|$ très petit pour éviter que plusieurs sauts se produisent sur chaque pas de temps (sinon, cela introduit un biais). Par ailleurs, nous avons observé numériquement que la variance de notre méthode explose quand on augmente trop λ et p.

- L'inconvénient majeur de notre méthode est qu'elle est très *coûteuse en temps de calcul*: elle nécessite un maillage en temps très fin et un grand nombre de trajectoires Monte Carlo (au moins 20 millions).

- Enfin, nous introduisons une méthode numérique d'évaluation d'actifs de stockage de gaz de type Monte Carlo pure, qui permet en particulier de traiter la variable de niveau de stock qui, pour rappel, est *dégénérée, contrôlée et contrainte*. Pour traiter les contraintes de volume, nous modifions la dynamique du niveau de stock de sorte que la contrainte est intrinsèquement satisfaite. Nous observons numériquement que les trajectoires du niveau de stock se resserrent quand l'intensité du processus de comptage augmente si le régime d'opération artificiel (processus à sauts purs) prend des valeurs équiprobables. En particulier, l'ensemble des niveaux de stock peut ne pas être décrit dans son ensemble par les trajectoires de niveaux de stock induites. Pour obtenir une méthode d'évaluation efficace, il nous semble donc nécessaire d'introduire un *choix approprié* de l'intensité de la mesure de sauts. Nous proposons une modification de la méthode permettant d'adapter l'intensité au comportement optimal *a priori* de l'actif de stockage.

0.2 Evaluation d'options sur moyenne mobile

0.2.1 Options Swing sur le marché du gaz

Un autre type de contrats est très commun dans l'industrie du gaz naturel depuis de nombreuses années: il s'agit de contrats d'approvisionnement dits contrats (ou options) Swing. Ces contrats ont été conçus pour permettre une flexibilité non seulement relative aux dates de livraison du gaz mais également aux volumes de gaz échangés. L'acheteur d'un tel contrat a le droit de recevoir des quantités plus ou moins importantes de gaz à certaines dates dans le futur contre un prix contractuel: l'acheteur peut soit augmenter ("swing up") ou diminuer ("swing down") le volume de gaz qu'il achète, d'où le nom d'option Swing.

En plus de différentes clauses incluant notamment des contraintes (dont des contraintes maximales et minimales sur le volume global de gaz acheté), des pénalités ou des droits supplémentaires, le vendeur du contrat spécifie le *prix du contrat* (terminologie utilisée en pratique), qui correspond en termes financiers au prix d'exercice auquel la quantité de gaz peut être achetée. Typiquement, ce prix d'exercice est indexé sur des moyennes mobiles de divers prix pétroliers: par exemple des moyennes sur les 6 derniers mois des prix du gasoil et du fuel, avec un délai en temps (ou retard) d'un mois.

Ce type d'options Swing est complexe car elles ne font pas seulement intervenir des exercices anticipés mais sont également *path-dependent*: le payoff de ces options dépend, par le biais du prix d'exercice, du prix de plusieurs matières premières sur toute la période (passée) de moyennisation. A temps continu, le problème est donc de *dimension infinie*. Dans un cadre à temps discret, la difficulté reste la *grande dimension* induite: la dimension du problème est égale au nombre de pas de temps de la fenêtre de moyennisation (à multiplier par le nombre de matières premières qui interviennent dans l'index, s'il y en a plusieurs).

Cette caractéristique spécifique du prix d'exercice des options Swing est très peu discutée dans la littérature et à notre connaissance n'a jamais été traitée, voir entre autres Bardou et al. [1] et Barrera-Esteve et al. [2]. Même dans le cas d'options à un seul exercice anticipé (options américaines), peu de littérature existe au sujet des options sur moyenne mobile. Certains utilisent des méthodes heuristiques (par exemple Bilger [3]) tandis que d'autres sont limités numériquement à des applications où la fenêtre de moyennisation est petite (par exemple Grau [15] et Kao et Lyuu [19]).

L'approche classique des praticiens est de supposer ce prix d'exercice *déterministe* (c'est-à-dire, exogène au système stochastique). Il y a donc potentiellement une importante perte de valeur à cause de cette approximation.

Une autre approche commune (voir e.g., Broadie et Cao [6]) est d'utiliser une approximation dite non Markovienne. Celle-ci consiste à calculer les estimateurs des espérances conditionnelles qui interviennent dans la procédure rétrograde de résolution (principe de programmation dynamique) en utilisant seulement deux variables explicatives: le prix spot du gaz et la moyenne mobile qui apparaît dans le payoff à la date considérée. Pourtant, aucun résultat (ni théorique, ni numérique) ne justifie cette approximation et il en résulte un prix d'option sous-optimal.

Une question plus générale posée par le problème présenté ci-dessus concerne l'*approximation des processus de moyenne mobile*. Notre premier objectif était donc de proposer une approximation en dimension finie de la dynamique de dimension infinie des processus de moyenne mobile. Avec une telle approximation et à temps continu, le problème de l'évaluation d'options

sur moyenne mobile (de dimension infinie) se ramène à un problème (Markovien) en dimension finie.

Dans les deux paragraphes suivants, nous donnons une intuition d'une telle approximation dans un cadre plus simple (options américaines) nous présentons les principaux résultats théoriques obtenus et résumons les conclusions des tests numériques que nous avons fait pour l'évaluation de différents types d'options sur moyenne mobile.

0.2.2 Une nouvelle méthode d'évaluation d'options sur moyenne mobile

Considérons pour simplifier un cadre mono sous-jacent. Les résultats présentés ci-dessous sont directement généralisables à des modèles multi sous-jacents ou à des modèles à facteurs de risque non observables (par exemple, volatilité stochastique). Par ailleurs, la même méthodologie peut être utilisée pour l'évaluation d'options à exercices anticipés multiples.

On note S le prix de l'actif sous-jacent (processus Markovien), X sa moyenne mobile sur une période de taille fixe $\delta > 0$ retardé en temps par un délai $l \geq 0$:

$$X_t = \frac{1}{\delta} \int_{t-l-\delta}^{t-l} S_u du, \quad \forall t \geq \delta + l, \tag{10}$$

et on considère le problème d'évaluation d'option américaine sur moyenne mobile:

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, X_\tau\right)\right]. \tag{11}$$

La dynamique du processus X est

$$dX_t = \tfrac{1}{\delta}\left(S_{t-l} - S_{t-l-\delta}\right) dt, \quad \forall t \geq \delta + l.$$

Cela montre en particulier que même si S est Markovien, le processus (S, X) ne l'est pas. La moyenne mobile X peut se réécrire plus généralement comme une intégrale ordinaire par rapport à une mesure de pondération μ sur $[0, +\infty)$, c'est-à-dire:

$$M_t = \int_0^\infty S_{t-u} \mu(du), \tag{12}$$

et on adopte la convention suivante pour les valeurs de S sur l'axe des temps négatifs: $S_t = S_0, \forall t \leq 0$. Dans le cas usuel (10), μ admet une densité uniforme:

$$\mu(dt) = h(t)dt, \quad h = \frac{1}{\delta} \mathbb{1}_{[l,l+\delta]}. \tag{13}$$

Nous aimerions fournir une approximation en dimension finie du processus de moyenne mobile M défini en (12): il s'agit donc de trouver n processus $X^0, \ldots, X^{n-1}$ tels que $(S, X^0, \ldots, X^{n-1})$ soit Markovien et M_t dépende de façon déterministe de $(S_t, X_t^0, \ldots, X_t^{n-1})$ d'une certaine manière. L'approximation que nous proposons est basée sur une approximation de la mesure de pondération μ comme un développement en série de fonctions de Laguerre tronquée à n termes. Cette technique est utilisée depuis longtemps en traitement du signal pour l'approximation de systèmes à dimension infinie (voir, e.g., Mäkilä [24]) mais est moins connue pour l'approximation des systèmes stochastiques.

La fonction $H(x) = \mu([x, +\infty))$ est approchée par un développement fini sur la base des fonctions de Laguerre *scalées* $(L_k^p)_{k\geq 0}$ ($p > 0$ est un paramètre d'échelle), qui forme une base orthonormale de L^2, c'est-à-dire[2]:

$$H_n^p(x) = \sum_{k=0}^{n-1} \langle H, L_k^p \rangle L_k^p(x),$$

où $\langle \cdot, \cdot \rangle$ désigne le produit scalaire L^2. Alors, en posant $h_n^p(t) = -\frac{d}{dt} H_n^p(t)$, on approche la valeur de la moyenne mobile M_t par

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \int_0^\infty S_{t-u} h_n^p(u) du, \quad \forall t \geq 0. \tag{14}$$

Une telle approximation (et en particulier, le coefficient de correction devant S_t) est choisie de telle sorte que la masse totale de la mesure de pondération de l'approximation de la moyenne mobile $M_t^{n,p}$ soit égale à la masse totale de la mesure de pondération μ de la moyenne mobile M_t. En particulier, elle devient exacte pour des prix sous-jacents S constants.

En fait, nous sommes passés du problème de l'approximation à dimension finie de M à celui de l'approximation de la mesure μ par une mesure $(H(0) - H_n^p(0))\delta_0(dt) + h_n^p(t)dt$. Nos résultats principaux sont les suivants:

- Nous introduisons les *processus de Laguerre* associés à notre approximation: il s'agit de n processus $X^{p,k}, k = 0, \ldots, n-1$ définis par

  $$X_0^{p,k} = S_0(-1)^k \frac{\sqrt{2p}}{p}, \quad X_t^{p,k} = \int_0^\infty L_k^p(u) S_{t-u} du, \quad \forall t \geq 0.$$

  Nous montrons que ceux-ci sont reliés à l'approximation de la moyenne mobile en (14) par

  $$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \sum_{k=0}^{n-1} a_k^p X_t^{p,k}, \quad \forall t \geq 0,$$

  pour des coefficients $a_k^p, k = 0, \ldots, n-1$ que l'on peut calculer explicitement.

- Nous justifions rigoureusement le fait que $(S, X^{p,0}, X^{p,1}, \ldots, X^{p,n-1})$ est un processus Markovien de sorte que le problème

  $$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \tag{15}$$

  constitue une approximation en dimension $(n+1)$ du problème (11).

- Nous démontrons un résultat général qui relie l'erreur forte d'approximation d'un processus de moyenne mobile par un autre, à une certaine distance entre leurs mesures de pondération. Avec les propriétés des fonctions de Laguerre, cela nous permet d'établir une borne sur l'erreur introduite en approchant M par $M^{n,p}$ quand n tend vers l'infini. Plus précisément,

[2]Dans ce paragraphe et contrairement à la section précédente, $\mathrm{L}^2 := \mathrm{L}^2([0, +\infty))$ désigne l'espace de Lebesgue des fonctions à valeurs réelles carré-intégrables définies sur $[0, +\infty)$ muni de sa norme usuelle.

si μ admet une densité qui satisfait certaines conditions appropriées (vérifiées par la densité en (13)),

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|M_t - M_t^{n,p}|\right] \leq C\varepsilon(n^{-\frac{3}{4}}), \tag{16}$$

où $\varepsilon(h)$, pour une taille de maille fixée h, désigne une borne supérieure du premier moment du module de continuité du processus S (voir Fisher et Nappo [14]).

- Dès que la fonction payoff ϕ est Lipschitz en sa seconde variable, cela nous permet d'obtenir directement une vitesse de convergence en $\varepsilon(n^{-\frac{3}{4}})$ pour l'erreur de *pricing* introduite quand on résout le problème approché (15) au lieu du problème initial (11).

- Dans le cas des moyennes mobiles uniformément pondérées, c'est-à-dire quand μ a une forme telle que (13), nous donnons des formules explicites pour les coefficients de Laguerre $\langle H, L_k^p\rangle$ et pour les coefficients a_k^p. La base de fonctions de Laguerre scalée de façon optimale fournit la meilleure vitesse de convergence en n de notre approximation. Ce paramètre de *scaling* optimal p_{opt} est unique pour un n donné et l'erreur dans L^2 qui en résulte

$$\left\|H - H_n^{p_{\text{opt}}(n)}\right\|_2$$

est invariante par changement d'échelle par rapport à la taille de la fenêtre de moyennisation δ et au délai l.

0.2.3 Résultats obtenus par l'approximation de Laguerre

D'un point de vue numérique, nous utilisons une approche de type *least squares* Monte Carlo pour résoudre le problème d'arrêt optimal approché (15), voir e.g. Bouchard et Warin [5]. Notre algorithme permet de traiter des problèmes où le vecteur d'état a une dimension maximale égale à 8 (c'est-à-dire $n = 7$ fonctions de Laguerre pour un modèle mono sous-jacent). Quand la fenêtre de moyennisation et/ou le délai est trop grand, il n'y a pas de méthode de référence (problème de la dimension). Nous comparons donc notre approximation au prix sous-optimal donné par l'approximation non Markovienne décrite plus haut, pour laquelle nous utilisons également une approche de type Longstaff et Schwartz [21]. Nous nous sommes restreints à des problèmes d'options américaines (un seul exercice anticipé) car les temps de calcul sont déjà très longs. Cependant, on peut s'attendre à retrouver *a fortiori* le même comportement de notre approximation dans le cas multi-exercice.

Les tests numériques que nous avons fait sur différents types d'options sur moyenne mobile démontrent l'efficacité et la flexibilité de notre méthode. En particulier, elle permet de traiter des cas où les périodes de moyennisation sont très grandes sans pour autant perdre en précision et les parties des algorithmes liées à l'approximation de Laguerre sont indépendantes du modèle de prix sous-jacent. Nous présentons ci-après les principales conclusions de notre étude numérique.

- Notre approche est très efficace pour approcher des processus de moyenne mobile (quand on utilise des bases de fonctions de Laguerre *scalées* de façon optimale). Il suffit d'environ $n = 3$ fonctions de Laguerre dans le développement (pour un délai nul $l = 0$) et de $n = 5$ fonctions (quand $l > 0$) pour obtenir une approximation très précise des dynamiques. Nous avons observé cela pour tous les modèles de prix que nous avons considérés et quelque soient la taille de la fenêtre de moyennisation et la valeur du retard en temps.

- Notre approximation de Laguerre fournit des résultats d'évaluation stables et convergés avec seulement $n = 3$ fonctions de Laguerre. Cela nous a permis par ailleurs de procéder à une étude plus générale du prix des options sur moyenne mobile, comme fonction de la période de moyennisation et du délai en temps.

- En comparant nos résultats avec ceux obtenus par la méthode d'approximation dite non Markovienne, nous avons observés que pour des options sur moyenne mobile standards ($l = 0$) l'erreur commise n'est pas très importante (moins d'1% dans les exemples que nous avons pris). Cela justifie l'utilisation en pratique d'une telle approche malgré sa sous-optimalité. Par contre, dans le cas d'options sur moyenne mobile avec un délai, cette approximation sous-optimale peut entraîner un large biais du prix de l'option (supérieur à 10% dans les tests que nous avons effectués).

- Pour l'évaluation d'une même option Bermudéenne sur moyenne mobile, nous avons observé numériquement que notre méthode converge beaucoup plus vite qu'une méthode de référence classique (en calculant les estimateurs des espérances conditionnelles par rapport au vecteur d'état composé des valeurs des prix du sous-jacent à chaque pas de temps de la fenêtre de moyennisation) relativement à la dimension de la variable d'état.

- Nous avons utilisé notre méthode pour évaluer des contrats d'approvisionnement sur le gaz de style Bermudéen incluant des caractéristiques réalistes: nous avons considérés des prix d'exercice, fonctions de moyennes mobiles de prix du gasoil et du fuel. L'approximation de Laguerre reste très précise pour approcher ces prix d'exercice indexés. Et pourtant, sur tous les exemples que nous avons pris, notre méthode donne des valeurs de contrats très proches (moins d'1% au-dessus) de ceux fournis par la méthode non Markovienne utilisée communément en pratique. Cela nous semble être dû à plusieures choses: ces prix d'exercice sont en fait mis à jour à dates fixes, typiquement tous les mois (et non pas tous les jours), ils font intervenir des périodes de moyennisation très longues (jusqu'à 6 mois pour des contrats d'un an) et des décalage en temps plutôt courts et enfin, les prix gaziers et pétroliers ont un comportement de retour à la moyenne qui lisse certainement les prix de ces contrats.

0.3 Organisation de la thèse

Cette thèse est composée de trois parties. L'évaluation des contrats de stockage et des options Swing a été sujet à de nombreux travaux ces dernières années. Dans une première partie, nous proposons une revue relativement exhaustive de la littérature, mettant en perspective les différentes approches d'évaluation existantes. Des considérations d'un point de vue autant théorique que numérique nous permettent de motiver les deux problématiques étudiées dans les deux parties suivantes. Ces parties II et III traitent des deux questions distinctes résumées dans cette introduction et peuvent être lues de manière indépendante.

La Partie II est associée à un papier intitulé *Swing options valuation: a BSDE with constrained jumps approach*, écrit en collaboration avec Huyên Pham, Peter Tankov et Xavier Warin, soumis à publication dans le livre *Numerical Methods in Finance*, Springer, qui sera édité en 2011. Nous y fournissons plus de détails sur les estimations des erreurs de pénalisation et de discrétisation dans les chapitres 3 et 4 respectivement. Dans le chapitre 5, des cas d'applications supplémentaires de la méthodes sont traités et les algorithmes numériques de résolution y sont

plus détaillés.

La Partie III est basée sur un papier intitulé *A finite dimensional approximation for pricing moving average options*, écrit en collaboration avec Peter Tankov et Xavier Warin, soumis à publication dans SIAM Journal on Financial Mathematics. Dans le chapitre 4, nous y donnons plus de détails sur les méthodes numériques d'évaluation d'options sur moyenne mobile introduites et incluons dans le chapitre 5 des résultats numériques sur des contrats gaziers indexés sur prix pétroliers.

Part

General introduction

The work presented in this PhD dissertation deals with methods and numerical issues for pricing gas contracts, and especially two kinds of contracts commonly traded in the gas market: gas storage and gas supplying contracts. Both of them include optionality and constraints, which make tough their valuation in a context of uncertain commodity prices. These questions naturally arise in the present European deregulated gas market and were motivated by practical challenges encountered by Électricité de France[3], main French electricity provider, which has diversified its activites to gas distribution in the past decade.

From a mathematical point of view, the valuation of these kinds of contracts (real options) leads to particularly complex stochastic control problems. These are multiple-obstacle problems, optimal switching problems or more generally impulse control problems. Besides, the specific gas contracts that we study involve additional difficulties both from a theoretical and numerical viewpoint.

On one hand, we identified non trivial constraints involved in gas storage contracts due to the operational characteristics of storage facilities. Gas storage valuation leads to a 3-regimes switching problem which involves a degenerate, controlled and constrained state variable (namely, the inventory level of the storage). The methods classically used require large computing times: a major challenge is thus to propose new pricing techniques with a smaller numerical cost. On the other hand, gas supplying contracts are known as indexed Swing options. The payoff at exercise involves an index which depends on moving averages of various commodity prices (gas oil, fuel oil, etc.): this leads to an infinite-dimensional pricing problem (in continuous time). In discrete time, there is a numerical challenge due to high dimensionality.

From those two problems, we derived two different problematic, holding both in a more general mathematical framework. Firstly, we introduce a numerical method for solving impulse control problems by using their representations as Backward Stochastic Differential Equations (BSDEs for short): these are BSDEs with constrained jumps. Secondly, we propose an approach for pricing moving average options, based on a finite-dimensional approximation of the infinite-dimensional dynamics of moving average processes.

In this dissertation, we thus introduce mainly two new resolution methods and in both cases, our approximation is justified by including an explicit rate of convergence of the approximate solution to the exact solution.

In the rest of this introduction we deal with the two distinct research questions mentioned above. We briefly describe the gas contracts considered, highlight the main challenges that we identified and then sum up our contributions.

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0.1 A BSDE-based method for solving impulse control problems

0.1.1 The particular case of gas storage contracts

In a real options framework, the no-arbitrage value of a gas storage facility can be viewed as the maximum expected revenues its holder can obtain by operating the facility optimally: the storage valuation problem is modeled from the perspective of the storage operator, which has to manage the storage facility in order to maximize its profits.

At each date before the expiration of the contract, the storage holder observes the market gas spot price and then determines the optimal strategy between three possible actions: inject, withdraw or do nothing. The storage is said to have three operating regimes (or modes). The storage cavity is linked to the delivery points via a network of pipelines. If the strategy is to inject, gas is purchased in the spot market and injected into the storage facility. If the strategy is to withdraw, gas is withdrawn and sold in the spot market. All these operations are subject to volumetric constraints, physical operational constraints such as maximum injection and withdrawal rates and various costs (in particular when switching from one mode to another).

The payoff implied by holding a storage facility is linear in the injection and withdrawal rates. In consequence, one can show easily that the optimal strategy is of a "bang-bang" type (in a continuous-time setting). In other words, it is always optimal to inject or withdraw at maximum possible rates. Thus, the continuous-time stochastic control problem of storage valuation boils down to a 3-*regimes optimal switching problem*.

The command corresponds to the regime decision and the state variable is composed of the gas spot price (or the various factors of the price model, if many) and the storage inventory level. This latter variable has three characteristics which make the optimal switching problem more complex both from a theoretical and numerial viewpoint: it is *controlled* by the regime decision (the stock level increases when injecting gas, decreases when withdrawing gas), *constrained* (due to volumetric constraints imposed in the cavity) and *degenerate* in the sense that its diffusion coefficient is zero.

To overcome the difficulty raised by the stock variable, one can use a discretization of the set of admissible inventory levels, see among others Barrera-Esteve et al. [2] and Warin [27]. Such an approach might be particularly memory-demanding since it is necessary to compute and keep in memory the solution at each inventory grid point. From a numerical viewpoint, Makassikis et al. [23] achieve large speed-up for storage valuations by using a parallelization algorithm. Besides, such a resolution on a *fixed and global* discretization grid leads to a numerical efficiency loss, since the exploration of some regions of the stock level might be useless to determine the optimal solution (the regions which are not crossed by the optimal stock level).

An other approach is possible: Carmona and Ludkovski [8] propose a bivariate least squares Monte Carlo scheme, which approximate conditional expectations involved in the backward recursion by a bivariate regression against the current gas price and the inventory level. This approach requires the simulation of inventory level paths, in other words to guess the current inventory level at each time step of the backward algorithm: in [8], the authors combine randomization and guesses of the optimal strategy. This constitutes an heuristics-based approximation, whose theoretical justification seems difficult. In the spirit of Carmona and Ludkovski [8], to avoid the inventory levels discretization, we would like to provide an alternative purely simulation-based numerical method using sample paths of gas price and inventory level.

We should point out that optimal switching can be viewed as a particular case of *impulse control* in higher dimension, namely when adding to the system a pure jump process (standing for the current operating regime) controlled by the regime decision. Thus, these considerations more generally raise the question of finding alternative probabilistic methods for solving impulse control problems with (possibly) degenerate forward processes controlled by the impulse strategy. The recently developed theory of BSDEs and its application to such kinds of problems gives new possibilities going in this sense. From a numerical viewpoint, simulation-based techniques are the natural way to solve BSDEs.

0.1.2 Resolution of impulse control problems by using BSDEs

BSDEs provide alternative characterizations of the solutions to optimal switching problems and more generally to impulse control problems. BSDEs linked to such problems are particularly complex and this remains a relatively unexplored domain from a numerical viewpoint, due to the difficulties raised by the practical resolution of those BSDEs.

In the particular case of multiple optimal switching, these are multi-dimensional reflected BSDEs (with oblique-type reflections), see among others, Hu and Tang [18] and Hamadène and Zhang [17]. Chassagneux et al. [11] have recently introduced a discretely obliquely reflected numerical scheme to solve such BSDEs and obtain a convergence rate of order $|\pi|^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$, when the step of the time grid is equal to $|\pi|$. However, this result holds in a framework where the forward state variable is *uncontrolled*.

In the more general framework of impulse control, the associated BSDEs are BSDEs with constrained jumps, see Kharroubi et al. [20] and this representation still holds for optimal switching as highlighted by Elie and Kharroubi [13]. Besides, such a characterization allows degenerate and controlled dynamics of the forward state variable of the impulse control problem. The main difficulty for numerically approximating such BSDEs comes from the constraint, which concerns the jump component of their solution. In particular, these BSDEs do not a priori involve any Skorohod type minimality condition. Classical approaches by *projected* schemes (discretely reflected backward schemes) are thus irrelevant.

An alternative consists in using a *penalization procedure*: that is, introduce the constraint in the BSDE driver and penalize it with a parameter $p > 0$: when the constraint is fulfilled, this additional penalization term disappears, and otherwise penalizes the driver with an exploding factor. The solution to the resulting penalized BSDE with jumps is known to converge to the minimal solution of the BSDE with constrained jumps, see [20] and [13]. However, no convergence rate is available. This penalization approach can also be viewed as an alternative to classical inductive schemes based on an iteration on the number of interventions, see for example Carmona and Touzi [9] in the case of Swing options or Chancelier et al. [10] and Seydel [26] in a general framework of impulse control (non-probabilistic solving approaches).

From a practical viewpoint, only few authors have published numerical experiments involving BSDEs for solving such kinds of problems [25, 16, 22]. In all those references, the forward processes are again *uncontrolled*.

Porchet [25] uses a numerical projected scheme for solving the 2-dimensional reflected BSDE associated to a real option problem (valuation of a power plant with two modes). The case of optimal switching with 2 regimes simplifies the computation of the solution since the associated 2-

dimensional reflected BSDE (which implies a normal-type reflection) can be reduced to a single BSDE with two reflecting barriers (by considering the difference value process). Hamadène and Jeanblanc [16], dealing also with a starting-stopping problem, use a penalization procedure for solving such a BSDE with two reflecting barriers. In a specific framework (in particular: uncontrolled forward diffusion and constant switching costs), the authors obtain a convergence rate of the error between the penalized and the exact solution of order p^{-1}.

Finally, Ludkovski [22] solves a 3-regimes switching problem by considering a cascade of reflected BDSEs with one reflecting barrier since the author uses an iteration on the number of switches, see also Carmona and Ludkovski [7]. By numerical observations, the authors conjecture that the total error of their numerical method grows linearly in the maximum number of switches.

On the other hand, a discrete-time backward scheme for solving BSDEs with jumps (without constraint) has been introduced by Bouchard and Elie [4]. Under standard assumptions on the FBSDE coefficients, the authors obtain a convergence rate of order $|\pi|^{\frac{1}{2}}$. Besides, Elie [12] presents numerical experiments involving BSDEs with jumps: the author alludes to the critical role of the intensity λ of the jump measure.

In consequence of those observations, our objectives were mainly twofold. First, we would like to provide a convergence rate of the error due to the penalization procedure described above. This would enable us to obtain a global rate of convergence of the error between the solution of an impulse control problem and the approximation given by the numerical solution to the penalized BSDE. In addition, this convergence rate should be explicit with respect to the approximation parameters introduced: namely, the jump intensity λ, the penalization coefficient p and the time step $|\pi|$. Such an error estimation is essential for numerical purposes as well (understanding of the numerical impact of those parameters) and allows to adjust in practice the fineness of the time grid π in relation to (λ, p). Finally, the efficiency of such a method has to be tested for solving real options problems.

0.1.3 A penalization approach for solving BSDEs with constrained jumps

Let us consider the impulse control problem

$$v(t,x) = \sup_{u=(\tau_k)_{k\geq 1}\in\mathcal{U}_{(t,T]}} \mathbb{E}\left[g(X_T^{t,x,u}) + \int_t^T f(X_s^{t,x,u})ds + \sum_{\substack{k\geq 1 \\ t<\tau_k\leq T}} \kappa(X_{\tau_k^-}^{t,x,u}) \right]. \tag{17}$$

An impulse control $u = (\tau_k)_{k\geq 1}$ is an increasing sequence of stopping times and the controlled state variable X^u is a càdlàg process such that

$$X_t^u = X_0^u + \int_0^t b(X_s^u)ds + \int_0^t \sigma(X_s^u)dW_s + \sum_{\tau_k\leq t} \gamma(X_{\tau_k^-}^u), \quad \forall t \geq 0, \tag{18}$$

where W is a d-dimensional Brownian motion. Under relevant assumptions, Kharroubi et al. [20] show that the solution to problem (17) can be represented as the minimal solution to the BSDE with constrained jumps:

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s - \int_t^T V_s dN_s + \int_t^T dK_s, \ \forall 0 \leq t \leq T \\ -V_t \geq \kappa(X_{t^-}), \ \forall 0 \leq t \leq T \end{cases} \tag{19}$$

in which N is a Poisson process with jump intensity $\lambda > 0$ and X is an (uncontrolled) jump diffusion process with dynamics:

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t + \gamma(X_{t^-})dN_t. \tag{20}$$

The solution (Y, Z, V, K) is said to be *minimal* if it has the smallest component Y in the (infinite) class of solutions to (19). Let us mention that this feature of the solution seems untractable for numerical issues.

We thus introduce an approach for solving BSDE with constrained jumps (19), based on the penalization of the obstacle constraint. More precisely, given a penalization parameter $p > 0$, this leads to a simpler BSDE with jumps:

$$Y_t^p = g(X_T) + \int_t^T \left[f(X_s) + p\left(V_s^p + \kappa(X_{s^-})\right)^+ \lambda\right] ds - \int_t^T Z_s^p dW_s - \int_t^T V_s^p dN_s, \tag{21}$$

whose unique solution (Y^p, Z^p, V^p) tends to the minimal solution (Y, Z, V) to (19) as $p \to +\infty$. Given a regular time grid $\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}$, we then consider the following backward discrete-time scheme for numerically solving the penalized BSDE with jumps (19), see Bouchard and Elie [4]:

$$\begin{cases} \bar{Y}_{t_N}^{p,\pi} = g(X_{t_N}^\pi) \\ \forall t_n \in \pi, t_n < T : \\ \qquad \bar{V}_{t_n}^{p,\pi} = \frac{1}{\lambda \Delta t_{n+1}} \mathbb{E}\left[\bar{Y}_{t_{n+1}}^{p,\pi} \Delta \tilde{N}_{t_{n+1}} | \mathcal{F}_{t_n}\right] \\ \qquad \bar{Z}_{t_n}^{p,\pi} = \frac{1}{\Delta t_{n+1}} \mathbb{E}\left[\bar{Y}_{t_{n+1}}^{p,\pi} \Delta W_{t_{n+1}} | \mathcal{F}_{t_n}\right] \\ \qquad \bar{Y}_{t_n}^{p,\pi} = \mathbb{E}\left[\bar{Y}_{t_{n+1}}^{p,\pi} | \mathcal{F}_{t_n}\right] + \left[f(X_{t_n}^\pi) + \left(p\left(\bar{V}_{t_n}^{p,\pi} + \kappa(X_{t_n}^\pi)\right)^+ - \bar{V}_{t_n}^{p,\pi}\right)\lambda\right] \Delta t_{n+1} \end{cases} \tag{22}$$

where $\Delta t_{n+1} = t_{n+1} - t_n$, $\Delta W_{t_{n+1}}$ is the Brownian increment on $[t_n, t_{n+1}]$ and $\Delta \tilde{N}_{t_{n+1}}$ the compensated version of the Poisson increment $\Delta N_{t_{n+1}}$.

We provide in this thesis a global convergence rate of the error introduced by the numerical approximation described above, with respect to λ, p and the time step $|\pi| := \Delta t_{n+1}, \forall n < N$. Assuming relatively general assumptions on the FBSDE coefficients (Lipschitz continuity of g, f, κ, b, σ and γ and boundness of γ), our main results are the following:

- As soon as the (random) number of optimal intervention of the considered impulse control problem belongs to L^2, we show that the value function in (17) is $\frac{1}{2}$-Hölder with respect to time maturity. This result holds true in the more general class of impulse control problems considered in Kharroubi et al. [20].

- We derive a rate of convergence of the error due to penalization under additional appropriate assumptions. Denote by $(Y^{p,t,x}, Z^{p,t,x}, V^{p,t,x})$ the solution to (21) when $X \equiv (X_s^{t,x})_{t \le s \le T}$ is the solution starting at x in t to SDE (20). We use an explicit functional representation for $Y^{p,t,x}$ as an essential supremum over a family of probability measures which only impact the jump intensity of N. By a convenient change of measure, which forces the penalized solution to jump as soon as possible after that an optimal impulse occurs, together with the continuity argument of the value function in its maturity variable,

we prove that:

$$\mathcal{E}^p = \sup_{0\leq t\leq T} \left| v(t,x) - Y_t^{p,t,x} \right| \leq C \left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} \right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right).$$

- We carefully perform an estimation of the discretization error introduced by the numerical scheme (22) in terms of additional parameters λ and p, by using the same arguments as Bouchard and Elie [4]. Under standard assumptions, we hightlight that a necessary condition for a convergence of order $|\pi|^{\frac{1}{2}}$ of the backward discrete-time scheme is

$$|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right). \tag{23}$$

  By classical regularization and Malliavin differentiation arguments applied to penalized BSDEs with jumps, this enables us to provide an explicit estimate of the discretization error: it is shown to exponentially grow with (λp^2).

- This allows us to deduce a global convergence rate of the numerical approximation introduced by our penalization procedure. In particular:

$$\begin{aligned}\mathcal{E}^p + \mathcal{E}^\pi(Y^p) &= \sup_{0\leq t\leq T} \left| v(t,x) - Y_t^{p,t,x} \right| + \left(\max_{n<N-1} \sup_{t_n\leq t\leq t_{n+1}} \mathbb{E}\left| Y_t^p - \bar{Y}_{t_n}^{p,\pi} \right|^2 \right)^{\frac{1}{2}} \\ &= \mathcal{O}_{p\to+\infty}\left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} + (1+\lambda)^2 \lambda p \bar{C}^{\lambda p^2} |\pi|^{\frac{1}{2}} \right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right) \end{aligned} \tag{24}$$

  for some constant $\bar{C}$ greater than 1 which does not depend either on λ, p, $|\pi|$ or α. Thus, for a sufficiently small time step $|\pi|$ with respect to λ and p, the global error is such that

$$[\mathcal{E}^p + \mathcal{E}^\pi(Y^p)]^* = \mathcal{O}_{p\to+\infty}\left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} \right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right). \tag{25}$$

Our second contribution concerns the practical use of such a numerical method. Combining the approach presented above with Monte Carlo techniques, we perform numerical experiments on some practical cases of impulse control in the framework of real options. We besides provide fully implementable algorithms. The results that we obtain allow a better understanding of the impact of the jump intensity and the penalization parameter from a numerical viewpoint:

- We first consider an optimal forest management problem which constitutes a non degenerate impulse control problem. We numerically retrieve the behavior of the approximation error as function of the jump intensity λ at fixed p, see (25). Accurate results are obtained (a quasi-analytical solution is available for this problem) and the method is efficient in this example.

- We handle with the problem of pricing multi-exercise options (normalized Swing options). This multiple optimal stopping time problem leads to a particularly degenerate three-dimensional impulse control problem. The method is clearly less competitive but we have been able (for small maximal number of exercises rights) to obtain stable and accurate valuation results, when increasing sufficiently the number of time steps and using enough Monte Carlo sample paths.

- We find that the numerical method is more sensitive to p than to λ, according to (23) and (24). In practice, the penalization parameter need to be chosen relatively small (less than 5 in the examples with took) and the time step $|\pi|$ very small to avoid multiple jump times on each time step (otherwise, this introduces a bias). Besides, we numerically observe that the variance of our method explodes when increasing too much λ and p.

- It appears that the main drawback of our method is its *slow computation speed*: it requires a very fine time grid and a large number of Monte Carlo sample paths (at least 20 million).

- Finally, we introduce a purely simulation-based method for pricing gas storage facilities which allows in particular to handle with the *degenerate, controlled and constrained* inventory level variable. To deal with the volumetric constraints, we modify the inventory level dynamics so that the constraint is intrinsically fulfilled. We observe that the trajectories of the stock level narrow when increasing the intensity of the counting process, if the simulated artificial operation regime (pure jump process) takes equiprobable values. In particular, the whole set of admissible inventory might not be crossed by resulting sample paths of the stock level. To obtain an efficient valuation method, it seems thus necessary to introduce a *suitable choice* of the intensity measure. We propose a modification of the numerical method, by adapting the intensity measure to the *a priori* optimal behavior of the storage.

0.2 Valuation of moving average options

0.2.1 Swing options in the gas market

Another kind of contract which has been commonplace in the natural gas industry for many years are Swing contracts. These contracts have been designed to allow flexibility of delivery with respect to time decisions and the amount of gas used. They gives to their buyer the right to receive greater or smaller amounts of gas at certain dates in the future at a contractual price: the buyer can either "swing up" or "swing down" the volume of purchase gas, hence the name Swing option.

In addition to various clauses including constraints (among which, maximal and minimal constraints on the global volume of purchased gas), additional penalties or rights, the contract seller specifies the so-called *contract price*, corresponding in financial terms to the strike price at which the amount of gas can be purchased by the option buyer. Typically, this strike price is indexed on moving averages of various oil prices: for example averages over the last 6 months of gas oil and fuel oil prices, delayed with a fixed 1 month lag.

These kinds of Swing options are complex because they do not only incorporate a multiple-exercise feature, but involve *path dependency* as well: the payoff at exercise depends on the entire history of the price of some commodities over the averaging period. In a continuous-time framework, the problem is thus *infinite dimensional*. In a discrete-time setting, there is a computational challenge, due to *high dimensionality*: the dimension is equal to the number of time steps within the averaging window (times the number of commodities involved in the index, if many).

This specific characteristic of Swing options strike price has been less highlighted in the literature, and to our best knowledge never explored, see among others Bardou et al. [1] and Barrera-Esteve et al. [2]. Even in the case of options with only one early exercise feature

(American style options), only few authors handle with moving average options. Either heuristic methods are used (see e.g., Bilger [3]) or they are computationally limited to applications where the moving average window is small (see Grau [15] and Kao and Lyuu [19]).

The strike price is commonly assumed to be *deterministic* (namely, exogenous to the stochastic system) by practitioners and there is potentially a significant value loss due to this approximation. An other common approach (see e.g., Broadie and Cao [6]) is to use a so-called non Markovian approximation. It consists in computing the conditional expectations estimators in the backward solving procedure (dynamic programming principle) by using only two explanatory variables: the underlying gas price and the moving average appearing in the option payoff at the considered time. No theoretical nor numerical result justifies this approximation and the resulting price is suboptimal.

A more general question raised by the problem discussed above is that of the *approximation of moving average processes*. Our first objective was to propose a finite-dimensional approximation of the infinite-dimensional dynamics of moving average processes. With such an approximation and in a continuous time, the infinite-dimensional problem of moving average option pricing boils down to a (Markovian) finite-dimensional problem. In the two paragraphs below, we provide an intuition of such an approximation in a simpler framework (American style options), present our main theoretical results and sum up the conclusions of the numerical experiments performed for pricing various kinds of moving average options.

0.2.2 A new method for pricing moving average options

Let us consider for the sake of simplicity a single-asset framework. The following results are directly generalizable to a multi-asset model or to a model with unobservable risk factors such as stochastic volatility. Besides, the same methodology can be used for pricing multiple-exercise options.

We shall denote by S the price of the underlying asset (Markov process), by X its moving average over a time window with fixed length $\delta > 0$ delayed with a fixed time lag $l \geq 0$:

$$X_t = \frac{1}{\delta} \int_{t-l-\delta}^{t-l} S_u du, \quad \forall t \geq \delta + l, \tag{26}$$

and consider the problem of moving average American option pricing:

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, X_\tau\right)\right]. \tag{27}$$

The process X follows the dynamics

$$dX_t = \tfrac{1}{\delta}\left(S_{t-l} - S_{t-l-\delta}\right) dt, \quad \forall t \geq \delta + l.$$

This shows in particular that even if S is Markovian, the process (S, X) is not. The moving average X can be written more generally as an ordinary integral with respect to a weighting measure μ on $[0, +\infty)$, namely:

$$M_t = \int_0^\infty S_{t-u} \mu(du), \tag{28}$$

and the following convention is adopted for the values of S on the negative time-axis: $S_t = S_0, \forall t \leq 0$. In the usual case (26), μ admits an uniform density:

$$\mu(dt) = h(t)dt, \quad h = \frac{1}{\delta}\mathbb{1}_{[l,l+\delta]}. \tag{29}$$

We would like to find a finite-dimensional approximation to the moving average process M in (28): that is, n processes $X^0, \ldots, X^{n-1}$ such that $(S, X^0, \ldots, X^{n-1})$ are jointly Markov and M_t depends deterministically on $(S_t, X_t^0, \ldots, X_t^{n-1})$ in some way. The approximation that we introduce is based on an approximation of the weighting measure μ as an expansion in a series of Laguerre functions truncated at n terms. This technique has long been used in signal processing for the approximation of infinite-dimensional systems (see, e.g., Mäkilä [24]), but is less known in the context of approximation of stochastic systems.

We approximate $H(x) = \mu([x, +\infty))$ by a truncated expansion on the scaled Laguerre functions $(L_k^p)_{k\geq 0}$ ($p > 0$ is a scale parameter), which constitute an orthonormal basis of L^2, that is[4]:

$$H_n^p(x) = \sum_{k=0}^{n-1} \langle H, L_k^p \rangle L_k^p(x),$$

in which $\langle \cdot, \cdot \rangle$ denotes the inner product of L^2. Then, setting $h_n^p(t) = -\frac{d}{dt}H_n^p(t)$, we approximate the moving average value M_t by

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \int_0^\infty S_{t-u} h_n^p(u) du, \quad \forall t \geq 0. \tag{30}$$

Such an approximation (and in particular the correction coefficient in front of S_t) is chosen so that the total mass of the weighting measure of the approximate moving average $M_t^{n,p}$ is equal to the total mass of the weighting measure μ of the moving average M_t. In particular, it becomes exact for a constant asset price S.

In some sense, we have transferred the problem of finding a finite-dimensional approximation for M to that of finding an approximation of the form $(H(0) - H_n^p(0))\delta_0(dt) + h_n^p(t)dt$ for the measure μ. Our main results are the following:

- We introduce the *Laguerre processes* associated to our approximation: these are n processes $X^{p,k}, k = 0, \ldots, n-1$ defined by

  $$X_0^{p,k} = S_0(-1)^k \frac{\sqrt{2p}}{p}, \quad X_t^{p,k} = \int_0^\infty L_k^p(u) S_{t-u} du, \quad \forall t \geq 0.$$

  We show that there are linked to the moving average approximation in (30) by

  $$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \sum_{k=0}^{n-1} a_k^p X_t^{p,k}, \quad \forall t \geq 0,$$

  for some explicit coefficients $a_k^p, k = 0, \ldots, n-1$.

[4]Be aware of the fact that in this paragraph, $\mathrm{L}^2 := \mathrm{L}^2([0, +\infty))$ denotes the Lebesgue space of real-valued square-integrable functions on $[0, +\infty)$ endowed with its usual norm.

- The process $(S, X^{p,0}, X^{p,1}, \ldots, X^{p,n-1})$ is shown to be Markovian so that the problem

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \tag{31}$$

  constitutes a $(n+1)$-dimensional approximation to problem (27).

- We provide a general result which links the strong error of approximating one moving average process with another to a certain distance between their weighting measures. Together with the properties of Laguerre functions, this allows us to establish a bound on the error made when approximating M by $M^{n,p}$ as n goes to infinity. Specifically, if μ admits a density satisfying some appropriate conditions (verified by the density in (29)),

$$\mathbb{E}\left[\sup_{0 \le t \le T} |M_t - M_t^{n,p}|\right] \le C\varepsilon(n^{-\frac{3}{4}}), \tag{32}$$

  where $\varepsilon(h)$, for a fixed mesh size h, denotes an upper bound of the first moment of the modulus of continuity of the process being average S (see Fisher and Nappo [14]).

- As soon as the payoff function ϕ is Lipschitz in its second variable, this enables us to provide a convergence rate of order $\varepsilon(n^{-\frac{3}{4}})$ for the pricing error made when solving approximate problem (31) instead of problem (27).

- In the case of uniformly-weighted moving averages, namely when μ has a form like (29), we give explicit formulas for the Laguerre coefficients $\langle H, L_k^p \rangle$ and coefficients a_k^p. The *optimal scaled* Laguerre functions basis provides the best convergence rate on n of our approximation. This optimal scale parameter p_{opt} is unique for given n and the resulting L^2-error

$$\left\| H - H_n^{p_{\text{opt}}(n)} \right\|_2$$

  is scale invariant with respect to the averaging window δ and the time lag l.

0.2.3 Results obtained by the Laguerre-based approximation

From a numerical point of view, we introduce a least squares Monte Carlo approach to solve the approximate optimal stopping problem (31), see e.g. Bouchard and Warin [5]. Our algorithm allows us to deal with state vectors up to dimension 8 (that is, $n = 7$ Laguerre functions for a single-asset model). For large averaging windows and/or time lags, no benchmark is available (curse of dimensionality). We thus compare our approximation to the suboptimal approximate price given by the non Markovian approximation described above, computed using a Longstaff and Schwartz [21] approach. We restricted ourselves to the single-exercise case due to large computational times. However, one could expect that the same observations would be *a fortiori* retrieved in the multiple-exercise case.

Numerical experiments have been performed for different kinds of moving average-type options which demonstrate the efficiency and flexibility of our method. In particular, it allows to deal with large averaging windows without loosing in accuracy and the numerical issues linked to the Laguerre approximation are independent of the underlying price model. We present hereafter our the main results of our numerical study.

- The Laguerre-based approach for approximating moving average processes is very efficient (when using an optimal scaling of the Laguerre basis). Around $n = 3$ functions in the Laguerre series (for a zero time delay $l = 0$) and $n = 5$ functions (when $l > 0$) are sufficient to provide very accurate dynamics approximation. This holds in all the price models that we took and whatever the values of the averaging period and the time delay.

- Our Laguerre approximation-based pricing method provides stable and converged option prices already with around $n = 3$ Laguerre functions. This allows us to perform a more general study of moving average options price as function of the averaging window and the time delay.

- Comparing our results with the non Markovian approximate method, we find that for standard moving average options ($l = 0$) the error is not so large (less than 1% for the examples we took). This justifies the use of this approach for practical purposes in spite of its suboptimality. However, for moving average options with time delay, the suboptimal approximation may lead to a large bias of the option's price (more than 10% in our numerical tests).

- For pricing a same moving average Bermudan option, we observe that our method converges much faster than a classical method (when computing the conditional expectations estimators with respect to the state vector composed of the prices values at each time step of the averaging window) with respect to the state dimension.

- We use our method for pricing Bermudan-style contracts on gas, including realistic characteristics: we deal with strike prices, functions of moving averages of gas oil and fuel oil prices. The Laguerre expansion-based approach remains very accurate for approximating those stochastic indexed strikes. And yet, on the various practical examples we took, our method gives contract values very close (less than 1% above) to the non Markovian approximate method commonly used in practice. This is mainly due to monthly-updated strike prices, large averaging windows (up to 6 months for one year contracts) and relatively small time delays, and probably the mean-reverting behavior of gas and oil prices.

0.3 Organization of the thesis

This manuscript is composed of three parts. Since a lot of research has been achieved in the field of gas storage facilities and Swing options valuation in the past decade, we provide in Part I a relatively exhaustive review of the existing valuation methods. Various considerations allow us to motivate the research questions addressed in the two following parts both from a theoretical and numerical point of view. The second and third parts treat the two distinct problems presented above and can be read independently of each other.

Part II is related to a paper entitled *Swing options valuation: a BSDE with constrained jumps approach*, written in collaboration with Huyên Pham, Peter Tankov and Xavier Warin, submitted to publication in the book *Numerical Methods in Finance*, Springer, which will be edited in 2011. More details on the penalization and discretization errors estimation are respectively given in Chapter 3 and Chapter 4 and additional numerical applications and details for numerical algorithms are presented in Chapter 5.

Part III is based on a paper entitled *A finite dimensional approximation for pricing moving average options*, written in collaboration with Peter Tankov and Xavier Warin and submitted to SIAM Journal on Financial Mathematics. More details on the solving algorithms are given in Chapter 3 and additional numerical results are presented in Chapter 4, including in particular experiments for oil-indexed contracts commonly encountered in the gas market.

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Part I

Valuation methods of gas contracts: a review

Chapter 1

Gas storage facilities

1.1 General description

The European gas market has in the last decades experienced some major changes and in particular the demand for gas is growing and expected to continue its development. In France, for example, since the 70's, the gas domestic consumption has increased faster (4 % per year on average) that the consumption in other energies (1 %)[1]. In this context, gas storage facilities play a significant role.

Why using storage facilities ? There are three major reasons for the use of storage facilities. Firstly, they allow to ensure that natural gas will be easily accessible in response to higher demand. Most of gas supply is designed for the residential and industrial sectors (residential: 39 %, industrial: 38 %, and services: 16 % in France)[1]. During periods of high consumption (that is in winter and the first days of the week), the key requirement is high deliverability. And yet, because gas is extracted from underground fields (in France, gas is principally imported from Norway, Russia, Netherlands, Algeria)[1], the gas supply is relatively inflexible: it is limited by the capacity of the pipeline system which deliver natural gas to end users. Storage facilities, which are in addition most often situated closer to the consumers, are thus the only significant supply regulator and demand buffer.

Secondly, due to new regulation conditions, distribution compagnies have now the obligation to own storage facilities to secure supply in periods of high demand. Thirdly, gas storage facilities with high deliverability (high injection and withdrawal capacities) are used as an arbitrage mechanism, since they can quickly respond to changing gas prices. Energy marketer use storage to exploit market arbitrage opportunities with objective to maximize profitability.

Different kinds of storage Gas storage facilities are separated in two categories, according to two major kinds of use. *Base load* facilities (or "seasonal storage") have large volume capacities but low deliverability rates. They are thus used by gas suppliers to satisfy seasonal demand: gas at lower price is stored in summer and delivered in winter to consumers. *Peak load* facilities (or "fast storage") on the contrary are smaller reservoirs with high injection and withdrawal rates and can be used for arbitrage on short periods in time (intraweek trading) and to respond to short-term demand.

[1] Figures and Data 2010 issued from `www.developpement-durable.gouv.fr`.

The three major types of underground storage facilities are depleted fields, aquifers, and salt caverns. Each storage type has different physical characteristics, among which the maximum capacity and deliverability rates are the most important. The two first one belong to the categorie of base load facilities (around 10 months are required to totally fill and empty the cavities), the third one is a peak load facility (around 2 weeks required to totally fill and empty the salt cavern). Salt caverns, in particular, involve non trivial characteristics due to thermodynamic properties of such storage facilities (see e.g., Hagoort [60] and Karimi-Jafari [70]). Let us mention here another kind of storage, more and more developing, which is not underground: natural gas can be store as LNG (Liquefied Natural Gas) in storage tanks. This requires a proper infrastructure and withdrawal rates are known to be fast and costly (a regasification is necessary before the injection in pipelines).

Real options framework Deterministic traditional methods fail to accurately capture the economic value of gas storage facilities in present competitive energy market, which implies in particular gas prices with high volatility and spikes. Intrinsic value of storage facilities can be classically determined by gas forward price curves, seasonal price spread and deterministic characteristics of the physical storage. However, their extrinsic value is highly dependent of market conditions.

The classical approach that has been adopted is thus the financial approach of real options theory. The real option valuation framework borrows the ideas from classical financial option pricing theory and views a real asset or investment project as an option on the underlying cash flows. More effort is usually needed for solving a real options valuation problem than for pricing a financial option, since real options include not only optionality but as well constraints due to their physical/operational characteristics. Schwartz and Trigeorgis [102] and references therein shall give to the interested reader a recent review of this large research field.

The valuation of storage contracts is a challenging problem because both the physical and the financial aspects of storage need to be considered. Storage operations (injecting gas in and releasing gas from the cavity) are subject to various (and sometimes complex) physical and operational constraints and financial costs might be involved as well: for limiting the number of operations, charging the inventory level left in the storage at expiration of the contract or imposing penalties if the final inventory is below some required level. Besides, a firm investing in a storage facility/a storage arbitrager has to take into account the uncertainty of gas prices. We provide in Paragraph 1.4 a quick recall of multi-factors price models mainly used by practitioners.

In a real options framework, the storage valuation problem is modeled as a *stochastic control problem*. The classical perspective for valuing a storage contract is the one from the storage operator, which has to manage optimally the storage facility, since the no-arbitrage value of a storage can be viewed as the maximum (risk-neutral) expected revenues its holder can obtain by operating the facility optimally.

At each date before the expiration of the contract, the storage holder observes the market gas spot price and then determines the optimal strategy between three possible actions: inject, withdraw or do nothing. The storage is said to have three operating regimes (or modes). If the strategy is to inject, gas is purchased in the spot market and injected into the storage facility. If the strategy is to withdraw, gas is withdrawn and sold in the spot market. All these operations are subject to volumetric constraints (in particular, the storage size can be time-dependent because one may want to hire a portion of the cavity), physical operational constraints such as maximum injection and withdrawal rates and various costs.

Recently, a lot of research has been achieved in the field of gas storage valuation including various assumptions on constraints and costs. From a theoretical point of view, discrete-time as well as continuous-time formulation of the valuation problem are considered. Let us notice here that the discrete-time framework most often roughly simplify the analysis of the problem. In this context, it is difficult to propose a unified and exhaustive (for any kind of storage facility and contract) modelization of gas storage assets. We try however to propose here, in continuous time, a relatively general formulation of the storage valuation problem by considering typical payoffs and financial costs and incorporating realistic volumetric and operational constraints. These last characteristics were approved by practitioners (cf. Warin [110]).

1.2 A gas storage modelization

Since the payoff implied by holding a storage facility is commonly linear in the injection and withdrawal rates, the optimal strategy is of a "bang-bang" type. In other words, it is always optimal to inject or withdraw at maximum possible rates. A justification of this statement is provided in Remark 1.2.1. Thus, the continuous-time stochastic control problem of storage valuation boils down to an *optimal switching problem*, which is the point of view that is adopted in our presentation. This formulation is actually close to the one from Carmona and Ludkovski [26].

Switching control and state variables Let T, be the expiration time of the storage contract. In the following, we shall denote by $S = (S_t)_{t\geq 0}$ the gas spot price (see Paragraph 1.4 for more details on classically-used gas price models) which is a Markov process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$, the natural filtration generated by S.

The storage management strategy u is a $\mathbb{F}$-adapted càdlàg process. u_t denotes the operating regime at time t and is valued in $\mathcal{I} = \{-1, 0, +1\}$, whenever the decision is to inject gas in the storage ($u_t = +1$), to withdraw ($u_t = -1$) or to store ($u_t = 0$). As in the classical literature of optimal switching (see among others Pham et al. [86]), u can be written as

$$u_t = \sum_{k\geq 0} \xi_k \mathbb{1}_{[\tau_k, \tau_{k+1})}(t), \quad \forall 0 \leq t \leq T. \tag{1.1}$$

in which

- $(\tau_k)_{k\geq 1}$ is an increasing sequence $\mathbb{F}$-stopping times corresponding to the switching dates (by convention $\tau_0 = 0$),
- $(\xi_k)_{k\geq 1}$ is the sequence of operating regimes: $\forall k \geq 1$, ξ_k is a $\mathcal{F}_{\tau_k}$-measurable variable such that $\xi_k \in \mathcal{I}$ and $\xi_k \neq \xi_{k-1}$. The initial facility mode ξ_0 is deterministic.

When injecting or withdrawing gas, the operator impact the inventory level of the storage, C^u, which is submitted to time-dependent volumetric constraints:

$$0 \leq \underline{c}_t \leq C^u_t \leq \overline{c}_t, \quad \forall 0 \leq t \leq T. \tag{1.2}$$

where $\underline{c}$ and $\overline{c}$ are given deterministic functions. The injection and withdrawal rates of the storage facility are commonly functions of the inventory level:

$$q(C^u_t, u_t) = \begin{cases} q_{\text{inj}}(C^u_t) & \text{if } u_t = +1 \text{ (injection)} \\ 0 & \text{if } u_t = 0 \text{ (storage)} \\ -q_{\text{with}}(C^u_t) & \text{if } u_t = -1 \text{ (withdrawal)} \end{cases} \tag{1.3}$$

where $q_{\text{inj}} > 0$ and $q_{\text{with}} > 0$ denote the maximal injection rate and withdrawal rate respectively, see typical examples in Figure 1.1. Indeed, when the inventory level in the cavity is high, so does the cavity pressure which makes injections more difficult than withdrawals. The converse holds when the storage level is low. Other form of injection and withdrawal capacities based on pressure laws are formulated in Davison et al. [42] but stepwise functions (called "ratchets") are mainly used in the industry.

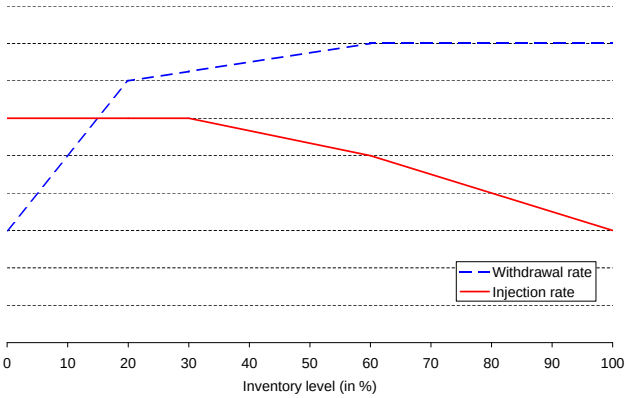

Figure 1.1: Inventory-dependent injection and withdrawal rates.

Finally, the inventory level satisfies by definition the ordinary differential equation

$$dC_t^u = q(C_t^u, u_t)dt \tag{1.4}$$

Payoff and financial costs At each date $t < T$, the profit of the storage holder is given by

$$f(S_t, C_t^u, u_t) = -q(C_t^u, u_t)\left(S_t + k(C_t^u, u_t)\right) \tag{1.5}$$

where

$$k(C_t^u, u_t) = \begin{cases} k_{\text{inj}}(C_t^u) & \text{if } u_t = +1 \text{ (injection)} \\ 0 & \text{if } u_t = 0 \text{ (storage)} \\ -k_{\text{with}}(C_t^u) & \text{if } u_t = -1 \text{ (withdrawal).} \end{cases}$$

$k_{\text{inj}} > 0$ and $k_{\text{with}} > 0$ respectively represents injection and withdrawal costs due to compressors operation, delivery charges, drying costs, etc.:

$$k_{\text{inj}}(c) = k_{\text{inj}}^0 + k_{\text{inj,min}}(c)\mathbb{1}_{\{c<C_{\text{pr}}\}} + k_{\text{inj,max}}(c)\mathbb{1}_{\{c\geq C_{\text{pr}}\}}$$
$$k_{\text{with}}(c) = k_{\text{with}}^0 + k_{\text{with,max}}(c)\mathbb{1}_{\{c<C_{\text{pr}}\}} + k_{\text{with,min}}(c)\mathbb{1}_{\{c\geq C_{\text{pr}}\}}$$

where C_{pr} denote some reference inventory level linked to the pressure of the pipeline network.

At the expiration of the contract, a pre-specified penalty payoff function is most often applied to the final inventory left in the storage:

$$g(S_T, C_T^u) = -\beta S_T C_T^u, \quad \text{for some constant } \beta > 0.$$

Finally, when switching from one regime to the other, fixed costs may be applied (switching operations facturation for example or costs when starting/stopping the compressors):

$$\kappa(i,j) > 0, \forall i \neq j \in \mathcal{I}^2, \quad \kappa(i,i) := 0, \forall i \in \mathcal{I}.$$

Given initial conditions at time t, $S_t = s$, $C_t^u = c \in [\underline{c}_t, \overline{c}_t]$ and an initial operating regime $u_t = i$, the storage value $v(t, s, c, i)$ is given by the expected cashflows that the storage manager can obtain when operating the storage facility in an optimal manner, that is

$$v(t, s, c, i) = \sup_{u \in \mathcal{U}_t^i} v(t, s, c, i; u), \tag{1.6}$$

where $\mathcal{U}_t^i$ is the set of above-described admissible switching strategies, starting in regime i at time t, and

$$v(t, s, c, i; u) = \mathbb{E}^{(t,s,c,i)} \left[\int_t^T f(S_r, C_r^u, u_r) dr - \sum_{\substack{k \geq 1 \\ t < \tau_k \leq T}} \kappa(u_{\tau_k^-}, u_{\tau_k}) + g(S_T, C_T^u) \right]$$

in which $\mathbb{E}^{(t,s,c,i)}[\cdot] := \mathbb{E}[\cdot | S_t = s, C_t^u = c, \tau_0 = t, \xi_0 = i]$.

Remark 1.2.1 (Bang-bang property of the optimal strategy)**.** The linear property of the payoff function (1.5) in the injection and withdrawal rates imply, in continuous time, that the optimal policy is necessarily of a "bang-bang" type. It means that it is always optimal to take one the three decisions: inject at maximum rate q_{inj}, withdraw at maximum rate q_{with} or do not intervene. In a continuous-time framework, structural properties of the optimal operation strategy of a gas storage facility have been recently largely documented under various assumptions, see Chen and Forsyth [33], Kaminski et al. [69], Lai et al. [74], Secomandi [103] and Wu et al. [115].

Let us consider zero switching costs and introduce the corresponding formulation of above-presented problem, when not restricting ourselves to bang-bang strategies. A strategy $q = (q_t)_{0 \leq t < T}$ is given by the amount of gas injected or withdrawn in the storage at each time before T. By convention $q_t > 0$ when injecting gas in the storage and q_t in the opposite case. It is subjected to physical operational constraints as previously introduced, that is

$$-q_{\text{with}}(C_t^q) \leq q_t \leq q_{\text{inj}}(C_t^q)$$

in which the inventory level, C_t^q satisfies

$$dC_t^q = q_t dt, \quad \underline{c}_t \leq C_t^q \leq \overline{c}_t.$$

With analog notations as before

$$f(S_t, C_t^q, q_t) = -q_t \left(S_t + k(C_t^q, q_t) \right)$$

where

$$k(C_t^q, q_t) = \begin{cases} k_{\text{inj}}(C_t^q) & \text{if } q_t > 0 \text{ (injection)} \\ 0 & \text{if } q_t = 0 \text{ (storage)} \\ -k_{\text{with}}(C_t^q) & \text{if } q_t < 0 \text{ (withdrawal)} \end{cases}$$

and the stochastic control problem is the following

$$v(t, s, c) = \sup_{q \in \mathcal{Q}_t} \mathbb{E}^{s,c} \left[\int_t^T f(S_r, C_r^q, q_r) ds + g(S_T, C_T^u) \right]$$

v is thus solution to the partial differential equation (PDE for short):

$$\mathcal{L}_s v(t,s,c) + \sup_{-q_{\text{with}}(c) \le q \le q_{\text{inj}}(c)} \{f(s,c,q) + qD_c v(t,s,c)\} = 0 \qquad \forall t < T \tag{1.7}$$
$$v(T,s,c) = g(s,c)$$

in which $\mathcal{L}_s$ is the infinitesimal operator associated to the price process S and D_s the first order derivative with respect to the inventory C. The optimization problem involved in equation (1.7):

$$\sup_{-q_{\text{with}}(c) \le q \le q_{\text{inj}}(c)} \{-q(s + k(c,q)) + qD_c v(t,s,c)\}$$

exhibits a bang-bang control feature, see Øksendal and Sulem [94]. Namely, if

$$\begin{aligned}
\mathbf{I} &= \{(t,s,c), D_c v(t,s,c) > s + k_{\text{inj}}(c)\} \\
\mathbf{S} &= \{(t,s,c), s - k_{\text{with}}(c) \le D_c v(t,s,c) \le s + k_{\text{inj}}(c)\} \\
\mathbf{W} &= \{(t,s,c), D_c v(t,s,c) < s - k_{\text{with}}(c)\}
\end{aligned}$$

denote respectively the injection, storage and withdrawal regions, the optimal strategy has the form:

$$q^*(t,s,c) = \begin{cases} q_{\text{inj}}(c) & \text{if } (t,s,c) \in \mathbf{I} \\ 0 & \text{if } (t,s,c) \in \mathbf{S} \\ -q_{\text{with}}(c) & \text{if } (t,s,c) \in \mathbf{W}. \end{cases}$$

up to the volume capacities of the storage. It means actually that one can restrict ourselves, without loss of generality, to the sub-set of admissible strategies $q = (q_t)_{0 \le t < T}$ such that

$$q_t = q(t,s,c) \in \{-q_{\text{with}}(c);\ 0;\ q_{\text{inj}}(c)\}.$$

Remark 1.2.2 (Bang-bang strategies in discrete time)**.** The reasoning above is true in continuous time, but does not hold in a discrete-time framework. In particular, the optimal bang-bang strategy for the continuous-time problem may not be optimal for the same problem discretized in time and in inventory level. Chen and Forsyth [33] shows that in practice the error made is small when assuming such a bang-bang feature. In the simpler case of Swing options (constant local and global volume constraints), Bardou et al. [8] provides a necessary condition for the existence of a bang-bang optimal strategy, see in Paragraph 2.2.2 of next Chapter 2.

1.3 Main difficulties of classical valuation methods

1.3.1 Solving Hamilton-Jacobi-Bellman equation

When including switching costs, the Hamilton-Jacobi-Bellman equation associated to the storage valuation problem (1.6) boils down to a system of quasi-variational inequalities, see for example Pham [98], Chap. 5:

$$\begin{aligned}
v(T,s,c,i) &= g(s,c) \\
\min\{-\partial_t v(t,s,c,i) - \mathcal{L}_s v(t,s,c,i) - q(c,i)D_c v(t,s,c,i) - f(t,s,c,i)\ ; & \\
v(t,s,c,i) - \max_{j \in \mathcal{I}, j \ne i}\{v(t,s,c,j) - \kappa(i,j)\}\} &= 0 \quad \forall t < T
\end{aligned} \tag{1.8}$$

where $\mathcal{L}_s$ is the second order local operator of the spot gas price S and D_s the first order derivative with respect to the inventory C. No boundary conditions for the inventory level, valued in

$([\underline{c}_t, \overline{c}_t])_{t \leq T}$, are needed since by definition of the injection/withdrawal rate $q(c,i)$ in (1.3), the characteristics are outgoing in the C-direction at $C = \underline{c}_t$ and $C = \overline{c}_t$.

PDE (1.8) involves various difficulties, both from a theoretical and numerical point of view. It involves strong non-linearity and strong coupling (interconnected obstacle terms). Besides, due to the degenerate inventory dynamics, classical finite difference methods can suffer numerical instabilities, since this PDE is convection-dominated (lack of second order term $D^2_{cc}v$). We refer to Section 1.4 for a review of numerical PDE-based approaches explored in the literature.

1.3.2 Dynamic programming principle computation

The storage value defined in (1.6) is solution to dynamic programming equation

$$v(t,s,c,i) = \sup_{\tau \in \mathcal{T}_{(t,T]}} \mathbb{E}^{(t,s,c,i)} \left[\int_t^\tau f(S_r, C_r^u, i) dr \right.$$
$$\left. + \max_{j \in \mathcal{I}, j \neq i} \{ v(\tau, S_\tau, C_\tau^u, j) - \kappa(i,j) \} \mathbb{1}_{\{\tau < T\}} + g(S_T, C_T^u) \mathbb{1}_{\{\tau = T\}} \right],$$

where $\mathcal{T}_{(t,T]}$ denotes the set of $\mathbb{F}$-stopping times valued in $(t,T]$, see e.g., Carmona and Ludkovski [26]. The numerical resolution of problem (1.6) by tree-based and simulation-based method use the discrete-time version of such a dynamic programming principle, see Section 1.4 for a survey.

Consider a time grid $\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}$ with a regular step $\Delta t = t_{n+1} - t_n, \forall n < N$. The set of admissible strategies is restricted to switching times taking values in π. Then the backward recursion algorithm is

$$\begin{aligned} \text{Terminal condition :} \quad & v(T,s,c,i) = g(s,c) \\ \text{For } n = N-1, \ldots, 0 : \quad & v(t_n,s,c,i) = \max_{j \in \mathcal{I}} \{ f(s,c,j)\Delta t - \kappa(i,j) \\ & \qquad + \mathbb{E}^{(t_n,s,c,i)} \left[v(t_{n+1}, S_{t_{n+1}}, c + q(c,j)\Delta t, j) \right] \Big\} \end{aligned} \tag{1.9}$$

recall convention $\kappa(i,i) = 0$. This means again that at any time t_n before maturity, the storage operator choose the optimal strategy between not intervening (the facility keeps the same mode i) or switching to the most interesting regime j and pay $\kappa(i,j)$.

The inventory level depends on the policy choice. To overcome this difficulty in the backward recursion, one can use a discretization of the set of admissible inventory levels, that is a stock grid with step Δc such that

$$\mathcal{C} = \left\{ \underline{c}_{t_n} + k\Delta c, \forall k = 0, \ldots, \frac{\overline{c}_{t_n} - \underline{c}_{t_n}}{\Delta c}, \forall n = 1, \ldots, N \right\}$$

and solve the backward scheme (1.9) at each inventory grid point $c \in \mathcal{C}$. This algorithm is easily implementable, but to provide a good numerical accuracy, it requires a fine inventory grid. Since at each time step, it is necessary to compute and keep in memory the solution at each inventory grid point, such an approach might be particularly memory-demanding. Besides, such a resolution on a *fixed and global* discretization grid leads to a numerical efficiency loss (the same kind as in lattice schemes), since the exploration of some regions of the stock level might be useless to determine the optimal solution. In other words, the effective optimal strategy does not

cross over some (possibly large) stock levels, which could be avoided in the backward recursion scheme.

Carmona and Ludkovski [26] raise this problem too and propose an alternative (see in Section 1.4). In a same way, we have tried to propose a purely simulation-based method for valuing gas storage. Basically, such an approach is based on both simulations of the gas price and the (regime-dependent) inventory level. In addition to avoid stock discretization, it would have other major advantages such that: being independent of the gas price model and still competitive in high dimensions (allowing to include multi-factors model). We refer to Section 1.5 for the ideas that we have developed in this direction, implying in particular the use of Backward Stochastic Differential Equations.

1.4 Review of existing methods

Classical numerical techniques borrowed from option pricing can handle with the problem of storage valuation and have been extensively studied in the literature in the past decade. We provide here a relatively global survey of the main and most recent publications.

A quick recall of commonly used gas prices model

Gas spot price models commonly used by practitioners are issued from classical multi-factors model describing the evolution of the forward curves. If $F(t,T)$ corresponds to the price of an unitary amount of gas (in MWh) at time t for a delivery at time T, the gas forward curve is most often said to satisfy the following dynamic (see e.g., Clewton and Strickland [36]):

$$\frac{dF(t,T)}{F(t,T)} = \sum_{i=1}^{d} \sigma_i(t) e^{-a_i(T-t)} dW_t^i,$$

where $W^i, i = 1, \ldots, d$ are (correlated) Brownian motions, σ_i volatility coefficients and a_i mean-reverting coefficients. With such a modelization, the gas spot price S is defined as the limit of the future price, that is

$$S_t = \lim_{T \to t} F(t,T).$$

Most of the time, two-factors models are sufficient to describe the term structure of forward curves. The first short-term component describe swift changes in the forward curve, while the second component deals with long-term changes in the forward curve and describes structural changes in the gas market such that modifications of the global gas demand/supply. In such a case, the gas spot price is said to satisfy a two-dimensional mean-reverting gaussian model:

$$\begin{cases} S_t = F(0,t) e^{-\frac{1}{2}V_t + X_t^S + X_t^L}, 0 \leq t \leq T \\ dX_t^S = -a_S X_t^S dt + \sigma_S(t) dW_t^S, \quad X_0^S = 0 \\ dX_t^L = -a_S X_t^L dt + \sigma_L(t) dW_t^L, \quad X_0^L = 0 \end{cases} \tag{1.10}$$

in which X^S and X^L are two Ornstein-Uhlenbeck processes with mean-reverting and volatility coefficients respectively equal to equal to (a_S, σ_S) and (a_L, σ_L), with $V_t = \text{covar}(X_t^S, X_t^S)$.

More complex model with jumps are sometimes used to take into account the spikes of gas spot prices observed in the market (namely, Day Ahead forward prices). Typically, the short term factor W^S can be replaced by a Lévy process with a Normal Inverse Gaussian distribution, see e.g., Benth et al. [14].

Numerical PDE-based techniques

Chen and Forsyth [33], Davison et al. [42] and Kaminski et al. [69] use numerical partial differential equation techniques. The storage value is viewed as the solution to the Hamilton-Jacobi-Bellman equation associated to the stochastic control problem of storage valuation, which has a form like (1.7) (switching costs are not included in the various modelizations). It is a two-dimensional non-linear PDE, whose resolution using finite difference methods is difficult in general. Since equation (1.7) is hyperbolic in the inventory level C (there is no diffusion term in the C-direction), classical second order finite-difference schemes can suffer instabilities.

Chen and Forsyth [33] propose a fully implicit scheme based on a semi-Lagrangian approach, developed for a class of one-dimensional gas spot prices model including mean-reverting and seasonality. The timestepping schemes are then extended to handle price jumps and later for regime-switching models in Chen and Forsyth [34]. Davison et al. [42] consider a one-dimensional gas spot price model with spikes and the resulting non-linear partial-integro differential equation is solved by using total variation diminishing methods. Kaminski et al. [69] deal with the same kind of gas prices model and the PIDE is solved by a Crank-Nicolson finite difference scheme.

Main drawbacks of these PDE-based techniques are: a lack of genericity with respect to the gas price model, non trivial implementation of the timestepping schemes and the well-known curse of dimensionality. In particular, those techniques could not handle with gas prices model in dimension 2 or 3 (involving short-term, middle-term and long-term factors). Finally, taking into account switching costs (charges when changing from one operating mode to another) might further complexify the PDE associated to the problem: this lead to a system of quasi-variational inequalities, see (1.8).

Tree-based approaches

Tree-based methods are used by Manoliu [91], Parsons [97], Barrera-Esteve et al. [10] and Warin [110]. They apply tree-building techniques to handle with the gas price stochastic evolution and apply for storage valuation the multi-layer tree model developed by Jaillet et al. [67] for Swing options pricing (each layer corresponds to a discrete inventory level). The proposed methods use forest of binomial/recombining trinomial trees which can handle one-factor or two-factors (see Parsons [97] and Warin [110]) gas prices dynamics.

These methods are highly memory demanding since the solution has to be computed (and kept in memory) at any point of the discretized space of admissible inventory levels, and even more when increasing the dimension of the gas price dynamics. The required memory size is acceptable for one-factor gas models but when dealing with two-factors models, the construction of a bivariate recombining tree is a more difficult problem (cf. Parsons [97], Warin [110] and Haahtela [59]) and the required number of tree nodes leads very often to an unfeasible numerical computation.

Monte Carlo-based methods

As a consequence, the most commonly used approaches for storage valuation are simulation-based. de Jong and Walet [41], Barrera-Esteve et al. [10], Holland [64], Makassikis et al. [88], Boogert and de Jong [16], Carmona and Ludkovski [26] and Warin [111] adjust the least squares Monte Carlo simulation technique for American options (see Longstaff and Schwartz [82] and Clément et al. [35]) to value storage. They still use a discrete inventory grid and regression-based techniques are used to approximate the conditional expectations of the storage value at

each time step of the backward recursion, see (1.9). With this approximated expectations, the optimal strategy can be obtained by comparing the payoffs under the three possible strategies: injection, withdrawal or storage. The value of accumulated cashflows in each price Monte Carlo path is then updated with respect to the optimal strategy in the backward resolution. Let us notice that Warin [111] deal also with gas storage hedging: the author provides a numerical method, based on the technique of tangent processes, to compute the conditional delta for a storage.

An alternative approach is possible: the approximated expectation can be directly introduced into the backward recursion of the dynamic programming equation. This scheme issued from Tsitsiklis and Van Roy [109] is proposed by Ludkovski [83], but confirmed to be less efficient that the LS scheme. Zhou [118] develop a generalized Gaussian quadrature scheme for estimating conditional expectations involved in the backward dynamic programming equation (this quadrature method is equivalent to a recombining tree).

The major advantages of the above simulation-based approaches are that they are independent of the price model and still competitive for multi-factors model. However, their main drawback is their slow computation speed. The discretization of the admissible inventory levels space is also memory-consuming. Makassikis et al. [88] achieve large speed-up for storage valuations by using a parallelization algorithm.

Carmona and Ludkovski [26] further propose a bivariate least squares Monte Carlo scheme, which approximate conditional expectations involved in the backward recursion by a bivariate regression against the current gas price and inventory level. This method requires thus the simulation of inventory level paths, in other words to guess the current inventory level at each time step of the backward algorithm. In [26], inventory levels are generated backward in time by combining *randomization* and *guesses of the optimal strategy*. The numerical experiments that we performed in this direction (at time t inventory levels sampled independently and uniformly from $[\underline{c}_t, \bar{c}_t]$, recall (1.2)) provide encouraging results. This might probably be improved by heuristically choosing the inventory level with respect to the most probable optimal strategy at time t (for example, with respect to the trends of the initial forward curve). However, such a heuristics-based approach could difficultly be justified from a theoretical point of view: in particular, the convergence of the resulting discrete-time scheme to the exact continuous-time solution. That's why this research direction was aborted.

Heuristic methods dealing with complex operational constraints

Taking into account operational constraints as inventory-dependent withdrawal and injection rates (recall example the behavior of q_{inj} and q_{with} in Figure 1.1) add a computational difficulty. This is in particular memory-consuming for tree-based and simulation-based methods. Only few of authors incorporate level-dependent capacities: Chen and Forsyth [33], Davison et al. [42] and Kaminski et al. [69], Manoliu [91], Makassikis et al. [88], Boogert and de Jong [16] and Carmona and Ludkovski [26]. Otherwise and most often, withdrawal and injection rates are assumed constant to help simplify the valuation problem.

A recent literature aims at understanding the effect of complex stock-dependent injection and withdrawal capacities on the storage value, but roughly simplify the analysis by considering heuristic methods. Among others, Wu et al. [115] use the so-called rolling intrinsic management policy: in each period, one decides the optimal strategy by solving a *static* optimization problem involving only forward prices (this constitutes an *intrinsic* dynamic programming equation). Based on the work of Secomandi [103] and Lai et al. [74], Wu et al. [115] studies the effectiveness of this heuristic scheme by analyzing the structure of the optimal strategy. The authors conclude

in particular that the storage value loss due to the approximation by heuristic strategies may be smaller to the one due to approximation the level-dependent injection/withdrawal rates by constant rates.

1.5 Alternative approaches by using BSDEs

Considering the large panel of methods that have been developed recently for gas storage valuation, it seems difficult to propose a new and competitive approach. As already mentioned in Paragraph 1.3.2 and in the line of Carmona and Lukovksy [26], our objective was to provide an alternative simulation-based numerical method using sample paths of both gas price and inventory level, *to avoid inventory level discretization*. In addition, we would like to derive an analysis of convergence of the method with respect to main approximation parameters.

The recently developed theory of Backward Stochastic Differential Equations (BSDEs for short) and its application to optimal switching problems gives new possibilities going in this sense. These approaches provide alternative characterizations of the solution to optimal switching problems than the classical ones presented in Section 1.3 and the natural way to solve BSDEs are simulation-based techniques. However, this remains a relatively unexplored domain from a numerical point of view, due to the difficulties raised by the practical computation of the solution to BSDEs, and more specifically *BSDEs related to multiple obstacle problem*: that is multiple optimal stopping time, optimal switching and impulse control problems.

BSDEs linked to such problems are particularly complex BSDEs: these are reflected BSDEs (RBSDEs for short), see among others Bouchard and Touzi [20], Hu and Tang [65], Hamadène and Zhang [63], and BSDEs with constrained jumps, see Kharroubi et al. [71] and Elie and Kharroubi [49]. A topic that have been a very active area of research in recent years is however the proposition of efficient numerical discrete-time schemes allowing the resolution of such BSDEs, see e.g., Bouchard and Touzi [20], Bouchard and Chassagneux [18], Bouchard and Elie [19], Chassagneux et. al [32].

And yet, it appears that only few authors have published numerical experiments on the subject. To our knowledge, only Hamadène and Jeanblanc [61] and Porchet [99] provide numerical results issued from BSDE-based computation for obstacle problems resolution: respectively for a two regimes switching problem with an one-dimensional uncontrolled forward diffusion (starting-stopping problem) and for the valuation by utility indifference of a coal and fuel oil-fired power plant with two modes (uncontrolled two-dimensional forward process). The case with 2 regimes roughly simplifies the computation of the solution as the implied two-dimensional reflected BSDE can be reduced to a single BSDE with two reflecting barriers (by working on the difference value process).

Let us also notice that the Tsitsiklis and Van Roy-based numerical scheme used by Ludkovski [83] is equivalent to an algorithm for solving a discrete-time reflected BSDE. The author handle with a 3-regimes switching problem with an uncontrolled forward diffusion by solving a cascade of RBDSEs with one reflecting barrier (using an iteration on the number of remaining switches).

In the case of storage valuation, the main difficulties raised come from the fact that it boils down to a 3-regimes switching problem whose state variable (actually, the inventory level) is *controlled*, *constrained* and *degenerate* (C^u admits no diffusion term), see (1.2) and (1.4).

1.5.1 Using multi-dimensional reflected BSDEs

Hu and Tang [65] and Elie and Kharroubi [49] make the link between multiple (that is $|\mathcal{I}| \geq 3$) switching problems in which the dynamics of the forward process is *policy-dependent* and multi-dimensional reflected BSDEs, namely systems of $|\mathcal{I}|$ reflected BSDEs with interconnected obstacles (the reflection is said to be of *oblique type*). This follows from recent work of Hamadène and Jeanblanc [61] (in the special case of two regimes), Hamadène and Zhang [62, 63] and Djehiche et al. [45] dealing with uncontrolled forward diffusions.

This kind of multi-dimensional reflected BSDE, in the case of an *uncontrolled* forward diffusion, can be solve with purely simulation-based techniques by using the numerical scheme proposed by Chassagneux et. al [32]. The discretization error when solving the obliquely reflected BSDE on a time grid with step $|\pi|$ is shown to be of order $|\pi|^{\frac{1}{2}-\epsilon}, \forall \epsilon > 0$.

A first approach using reflected BSDEs

We have first tried to use the result from Hu and Tang [65]. A representation of the solution of an optimal switching problem based on a multi-dimensional reflected BSDE is provided under relevant assumptions, among which the forward controlled state variable X must have a dynamic like

$$dX_t^u = b(X_t^u, u_t)dt + \Sigma(X_t^u)dW_t$$

in which Σ is *invertible*. The BSDE representation of the (optimal) solution is indeed based on a (optimal) policy-dependent change of measure. In our framework of storage valuation, the two-dimensional state variable (P, C^u) is degenerate since the inventory level admits no diffusion term. To overcome this difficulty, we had in mind to introduce a small volatility coefficient $\epsilon > 0$ (aimed at tending to 0) and transform the inventory dynamic in (1.4) as

$$d\bar{C}_t^u = q(\bar{C}_t^u, u_t)dt + \epsilon d\bar{W}_t^C$$

in which W^C is a standard Brownian motion. Since the inventory level is required to satisfy the volumetric constraint (1.2), it seems natural to impose the same thing to $\bar{C}^u$. It means, to make $\bar{C}^u$ intrinsically satisfy the constraint thought a modified dynamic: this constitutes a classical practical technique when solving stochastic optimization problems involving controlled and constrained forward processes. Let us thus redefine

$$d\bar{C}_t^u = q(\bar{C}_t^u, u_t)dt + \epsilon d\bar{W}_t^C - dA_t^u, \quad C_0^u = c \tag{1.11}$$

in which $A^u = A^{u+} - A^{u-}$ is the unique finite variation process with $A_0^u = 0$ such that the couple (C^u, A^u) is solution to (1.11) with

$$\begin{array}{rcl} \int_0^T \left(\bar{C}_t^u - \underline{c}_t\right) dA_t^{u+} & = & 0, \\ \int_0^T \left(\bar{c}_t - \bar{C}_t^u\right) dA_t^{u-} & = & 0. \end{array}$$

so that the constraint (1.2) holds for $\bar{C}_t^u$. Denote then by $\bar{v}^\epsilon$ the solution to the same problem as (1.6) in which the inventory level C^u has been replaced by this modified version $\bar{C}^u$.

Assume for the sake of simplicity here that the gas price process S satisfies a one-factor gaussian model driven by a standard Brownian motion W^S and set $S_0 = s$. Then, by adapting the arguments of [65], this leads to consider the following (system of) reflected BSDEs:

$$\begin{cases} Y_t^{i,\epsilon} = g(S_T, C_T) + \int_t^T h^\epsilon(S_r, C_r, Z_r^{i,\epsilon}, i)ds + \int_t^T dK_s^{i,\epsilon} - \int_t^T Z_s^{i,\epsilon} \cdot dW_s \\ Y_t^{i,\epsilon} \geq \max_{j \neq i} \left\{ Y_t^{j,\epsilon} - \kappa(i,j) \right\} \\ \int_0^T (Y_t^{i,\epsilon} - \max_{j \neq i} \left\{ Y_t^{j,\epsilon} - \kappa(i,j) \right\}) dK_t^{i,\epsilon} = 0 \end{cases} \tag{1.12}$$

where

$$\begin{array}{rccl} h^\epsilon: & \mathbb{R}\times\mathbb{R}\times\mathbb{R}^2\times\mathcal{I} & \longmapsto & \mathbb{R} \\ & (s,c,z=(z_1,z_2)^\perp,i) & \longrightarrow & \psi(s,c,z,i)=f(s,c,i)+\frac{1}{\epsilon}q(c,i)z_2 \end{array}$$

$W=(W^S,W^C)^\perp$, W^C is a standard Brownian motion ($\langle W^S,W^C\rangle=0$), C is solution to

$$dC_t=\epsilon dW_t^C-dA_t,\quad C_0=c \tag{1.13}$$

and $A=A^+-A^-$ is the unique finite variation process with $A_0=0$ such that the couple (C,A) is solution to (1.13) with

$$\begin{array}{rcl} \int_0^T (C_t-\underline{c}_t)\,dA_t^+ &=& 0, \\ \int_0^T (\bar{c}_t-C_t)\,dA_t^- &=& 0. \end{array}$$

BSDE (1.12) constitues a multi-dimensional reflected BSDE with interconnected obstacles and the reflection is said to be of oblique type, since the multi-dimensional value process $(Y^{i,\epsilon})_{i\in\mathcal{I}}$ takes values in a convex domain and is obliquely reflected on the boundaries of the domain. The third relation of (1.12) is a Skorohod *minimality condition*: the process $K^{i,\epsilon}$ is the minimal process allowing the solution $Y^{i,\epsilon}$ to be above the obstacle.

Existence and uniqueness of a solution $(Y^{i,\epsilon},Z^{i,\epsilon},K^{i,\epsilon})_{i\in\mathcal{I}}$ to (1.12) is ensured and the adaptation of the results from [65] implies that:
(a) for any $u\in\mathcal{U}_0^i$,

$$\bar{v}^\epsilon(0,s,c,i;u)\le Y_0^{i,\epsilon}, \tag{1.14}$$

(b) there exists an optimal switching strategy $u^*=(\tau_k^*,\xi_k^*)_{k\ge1}\in\mathcal{U}_0^i$ with ($\tau_0^*=0$ and $\xi_0^*=i$) explicitly defined recursively as follows:

$$\tau_k^*=\inf\left\{t\ge\tau_{k-1}^*, Y_t^{\xi_{k-1}^*,\epsilon}=\max_{j\neq\xi_{k-1}^*}\left\{Y_t^{j,\epsilon}-\kappa(\xi_{k-1}^*,j)\right\}\right\},\quad \forall k\in\mathbb{N}^* \tag{1.15}$$

and ξ_k^* is the $\mathcal{F}_{\tau_k^*}$-measurable random variable such that

$$Y_{\tau_k^*}^{\xi_{k-1}^*}=Y_{\tau_k^*}^{\xi_k^*}-\kappa(\xi_{k-1}^*,\xi_k^*),\quad \forall k\in\mathbb{N}^* \tag{1.16}$$

providing

$$\bar{v}^\epsilon(0,s,c,i;u^*)=Y_0^{i,\epsilon}. \tag{1.17}$$

Solving BSDE (1.12) together with (1.13) is feasible by using the discretely obliquely reflected numerical scheme proposed by [32]. Let π be a discrete-time grid with N time steps. For numerical reasons, it is clearly easier to exploit (a) than (b). (b) would require the resolution of N RBSDE (1.12) (starting at each time grid point) to compute the (discrete-time) optimal strategy with (1.15)-(1.16) and finally get an approximation to $\bar{v}^\epsilon(0,s,c,i)$. On the contrary, our idea consisted in simply exploiting (a) to obtain an approximation of the value function at time 0, see (1.14). First promising results when solving (1.12) by a discrete-time reflected scheme were obtained. We however encountered some stability problems in the numerical resolution, since the driver h^ϵ explodes as ϵ goes to 0.

And yet, the following obvious remark made us abandon this approach: on the contrary to what may seem at first sight, the solution $\bar{v}^\epsilon$ does not tend to v as ϵ goes to 0 because of the modification performed in (1.11) on the policy-depend inventory level C^u to require the *satisfaction of the constraint*. Indeed and for example, when the inventory level is so high that

there is not enough room for injecting gas in the storage facility, initial problem (1.6) with solution v say that gas cannot be injected into the facility: the operator need to wait and then switch to withdrawal at the most appropriate future time. On the contrary, in modified problem with solution $\bar{v}^\epsilon$, the operator can still inject gas as the constraint (1.2) will always hold for $\bar{C}_t^u$, recall (1.11).

In fact, $\bar{C}^u$ behaves as the water level of a dam: when the dam is full, water can spill out of the dam even if rain precipitations occur. The same reasoning holds when the inventory level is too low for withdrawal. In other words, for a same strategy u, the dynamic of $\bar{C}^u$ cannot tend to the one of C^u as $\epsilon \to 0$ (which is however required for an same almost sure satisfaction of bound constraints). This can be viewed easily when regarding the PDEs associated to both solutions. Because of (1.11), boundary condition associated to solution $\bar{v}^\epsilon$ is of Neumann type:

$$D_c \bar{v}^\epsilon(t,s,c,i) = 0, \quad \forall (t,s,c,i) \in [0,T) \times \mathbb{R} \times \{\underline{c}_t, \bar{c}_t\} \times \mathcal{I}$$

Comparing to (1.8), v does not satisfy the same boundary condition.

In conclusion, this approach failed because of two main characteristics of the controlled forward process involved in the storage valuation problem: the inventory level admits no diffusion term and is constrained. However, such a method could be used for the valuation of hydro power plant, in which the water level in the dam admits a diffusion term because of the presence of (random) inflows. This should provide an alternative RBSDE-based approach to existing methods, see for example Porchet [99] using a PDE-based methodology.

Another representation using reflected BSDEs

A new characterization result of the solution to optimal switching problems which does not require the invertibility of the volatility matrix of the underlying diffusion has been very recently provided by Elie and Kharroubi [49], following previous work of Djehiche et al. [44]. This representation is linked to a family of reflected BSDEs depending on a starting time and a starting value for the forward process.

Let us introduce the following family of reflected BSDEs, defined for any couple (ν, η), with ν a $\mathbb{F}$-stopping time and η a random variable $\mathcal{F}_\nu$-mesurable:

$$\begin{cases} Y_t^{\nu,i,\eta} = g(X_T^{\nu,i,\eta}) + \int_t^T f(X_s^{\nu,i,\eta}, i)\mathbb{1}_{\{s \geq \nu\}} ds - \int_t^T Z_s^{\nu,i,\eta} \cdot dW_s + \int_t^T dK_s^{\nu,i,\eta} \\ Y_t^{\nu,i,\eta} \geq \mathcal{M}_t^{\nu,i,\eta} := \max_{j \in \mathcal{I}} \left\{ Y_t^{\nu,j,\eta} - \kappa(i,j) \right\} \\ \int_0^T \left(Y_t^{\nu,i,\eta} - \max_{j \in \mathcal{I}} \left\{ Y_t^{\nu,j,\eta} - \kappa(i,j) \right\} \right) dK_t^{\nu,i,\eta} = 0 \end{cases} \tag{1.18}$$

where $X^{\nu,i,\eta}$ is the forward diffusion:

$$\begin{aligned} X_t^{\nu,i,\eta} &= \eta \mathbb{1}_{\{t \geq \nu\}} + \int_0^t b(s, X_s^{\nu,i,\eta}, i)\mathbb{1}_{\{s \geq \nu\}} ds + \int_0^t \Sigma(X_s^{\nu,i,\eta})\mathbb{1}_{\{s \geq \nu\}} dW_s, \quad \forall t \geq 0 \\ &= \begin{cases} 0 & \text{if } t < \nu, \\ \eta + \int_\nu^t b(s, X_s^{\nu,i,\eta}, i) ds + \int_\nu^t \Sigma(X_s^{\nu,i,\eta}) dW_s & \text{if } t \geq \nu. \end{cases} \end{aligned} \tag{1.19}$$

The existence and uniqueness of a solution to (1.18) has been proved in Hamadène and Zhang [62] under relevant assumptions and in particular, it does not require the invertibility of Σ. This applies thus, up to some adaptations, to our storage valuation context. Recall that the gas spot price is an uncontrolled variable. Set $X^{\nu,i,\eta} := (S^{\nu,\eta^S}, C^{\nu,i,\eta^C})^\perp$ for any $i \in \mathcal{I}$, (η^S, η^C) being two

$\mathcal{F}_\nu$-measurable random variables. C_t^{ν,i,η^C} represents the inventory level at time t if the operating regime is i on $(\nu, t]$:

$$C_t^{\nu,i,\eta^C} = \left(\underline{c}_\nu \vee \eta^C \wedge \bar{c}_\nu\right) \mathbb{1}_{\{t \geq \nu\}} + \int_\nu^t \bar{q}(s, C_s^{\nu,i,\eta}, i) \mathbb{1}_{\{s \geq \nu\}} ds, \quad \forall t \geq 0,$$

and $\bar{q}$ is a modified version of the injection/withdrawal rate q which makes the inventory level satisfy the constraint (1.2), namely, recalling (1.3),

$$\bar{q}(t, c, j) = \begin{cases} +q_{\text{inj}}(c) \mathbb{1}_{\{c \leq \bar{c}_t\}} & \text{if } j = +1 \text{ (injection)} \\ 0 & \text{if } j = 0 \text{ (storage)} \\ -q_{\text{with}}(c) \mathbb{1}_{\{c \geq \underline{c}_t\}} & \text{if } j = -1 \text{ (withdrawal).} \end{cases} \tag{1.20}$$

Let us mention that this modification of the dynamic of the inventory level is not problematic here as no diffusion term is involved.

Elie and Kharroubi [49] provides a representation of the optimal solution to optimal switching problem (1.6) using the class of RBSDEs (1.18). Let us define the storage switching strategy $u^* = (\tau_k^*, \xi_k^*)_{k \geq 1}$ given by $(\tau_0^* = 0, \xi_0^* = i)$ and for any $k \in \mathbb{N}^*$:

$$\begin{cases} \tau_k^* = \inf \left\{ t \geq \tau_{k-1}^*, Y_t^{\tau_{k-1}^*, \xi_{k-1}^*, X_{\tau_{k-1}^*}^*} = \mathcal{M}_t^{\tau_{k-1}^*, \xi_{k-1}^*, X_{\tau_{k-1}^*}^*} \right\} \\ \xi_k^* \text{ is such that } \mathcal{M}_{\tau_k^*}^{\tau_{k-1}^*, \xi_{k-1}^*, X_{\tau_{k-1}^*}^*} = Y_{\tau_k^*}^{\tau_k^*, \xi_k^*, X_{\tau_k^*}^*} - \kappa(\xi_{k-1}^*, \xi_k^*) \end{cases} \tag{1.21}$$

where $X^* = (S^*, C^*)^\perp$ is the diffusion

$$X_t^* = x + \sum_{k \geq 0} \int_{\tau_k^*}^{\tau_{k+1}^*} b(s, X_s^*, \xi_k^*) \mathbb{1}_{\{s \leq t\}} ds + \sum_{k \geq 0} \int_{\tau_k^*}^{\tau_{k+1}^*} \Sigma(X_s^*) \mathbb{1}_{\{s \leq t\}} dW_s, \quad \forall t \geq 0.$$

in which $x := (s, c)^\perp$. Then, the strategy u^* is optimal for problem (1.6), that is

$$v(0, s, c, i) = v(0, s, c, i; u^*) = Y_0^{0,i,(s,c)}. \tag{1.22}$$

This constitutes a strong and interesting result from a theoretical point of view. However the use of this result seems unfeasible in practice since the computation of the optimal solution/value function at time 0 would require the resolution of an (infinite) class of RBSDE, that is to solve RBSDE (1.18) for (infinitely) many initial values $(\tau, \nu^S, \nu^C) \in [0, T] \times \mathbb{R} \times [\underline{c}, \bar{c}]$.

1.5.2 Using BSDEs with constrained jumps

As we have seen in the PDE formulation (1.8) or with the representation using multi-dimensional reflected BSDE in previous Paragraph (1.5.1), the difficulty in the derivation of a tractable BSDE representation for the storage valuation problem is firstly the dependence of the solution in the regime $i \in \mathcal{I}$ with respect to the global solution in all possible regimes, and secondly the regime-dependence of the forward process.

Elie and Kharroubi [49] make the link between RBSDEs and a new class of BSDEs with constrained jumps, previously introduced in Kharroubi et al. [71] in the framework of impulse control. These BSDEs with constrained jumps provide a representation for the solution of impulse

control and optimal switching problems, see Remark 1.5.1. In particular, in [49], the family of reflected BSDEs (1.18) is related to a particular member of the class of associated BSDEs with constrained jumps. The idea behind this new representation is to artificially introduce a random regime I which jumps from one operating mode to another. This allows to retrieve, in the jump component of resulting one-dimensional BSDE, the required information with respect to the whole set of operating regimes.

In addition, it allows degenerate forward processes (do not require the invertibility of the volatility matrix of the forward process). According to us, this constitues the right BSDE-based formulation for numerically solving the storage valuation problem or any other multiple optimal switching problems involving *degenerate and policy-dependent state variables*.

Assume again for the sake of simplicity here that the gas price process S satisfies a one-factor Gaussian model driven by a standard Brownian motion W and set $S_0 = s$. Let us introduce a Poisson random measure μ on $\mathbb{R}^+ \times \mathcal{I}$ independent of W, with intensity measure $\lambda(dj)dt$ for some finite intensity λ such that $\lambda(j) > 0, \forall j \in \mathcal{I}$. Introduce the following pure jump process I:

$$I_t = i + \int_0^t \int_{\mathcal{I}} (j - I_{r^-}) \mu(dr, dj).$$

For a better understanding, one can associated to μ a unique *marked point process* $(\mathrm{T}_k, \chi_k)_{k \geq 1}$ with $\mathrm{T}_0 = 0$ and $\chi_0 = i$ (the χ_k are $\{-1; 0; +1\}$-valued random variable) such that:

$$I_t = \sum_{k \geq 0} \chi_k \mathbb{1}_{[\mathrm{T}_k, \mathrm{T}_{k+1})}(t), \quad \forall t \geq 0. \tag{1.23}$$

Comparing the above definition to the one of a storage switching strategy u in (1.1), I provides an artificial mode of operation: switching at time T_k from mode χ_{k-1} to χ_k according to a random measure. Let the (uncontrolled) gas process S be unchanged and on the contrary construct the process C^I, which follows a similar dynamic as the controlled inventory level C^u, but is based on the random regime I (that is injection when $I_t = +1$, storage when $I_t = 0$ and withdrawal when $I_t = -1$):

$$dC_t^I = \bar{q}(t, C_t^I, I_t)dt, \quad C_0^I = c$$

where $\bar{q}$ is the modified version of the injection/withdrawal rate q which makes the variable C^I satisfy the same constraint (1.2) as C^u, that is (1.20). Then, consider the following BSDE with constrained jumps:

$$\begin{cases} Y_t = g(S_T, C_T^I) + \int_t^T f(S_r, C_r^I, I_r)dr - \int_t^T Z_r dW_r - \int_t^T \int_{\mathcal{I}} V_r(j)\mu(dr, dj) + \int_t^T dK_r \\ -V_t(j) + \kappa(I_{t^-}, j) \geq 0, \forall j \in \mathcal{I} \end{cases} \tag{1.24}$$

Under relevant assumptions (in particular, one might have to smoothen the coefficient $\bar{q}$ of the forward process C^I), the existence of minimal solution to (1.24) is ensured, see citeEK10. The solution is minimal in the sense, that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{K})$ to (1.24), then $Y_t \leq \tilde{Y}_t, \forall 0 \leq t \leq T$. In addition, [49] provides the following interpretation with respect to the family of RBSDEs (1.18):

$$\begin{aligned} \forall 0 \leq t \leq T, \quad Y_t &= Y_t^{t, I_t, S_t, C_t^I}, \\ Z_t &= Z_t^{t, I_{t^-}, S_t, C_t^I}, \\ V_t(j) &= Y_t^{t, j, S_t, C_t^I} - Y_{t^-}^{t, I_{t^-}, S_t, C_t^I}, \forall j \in \mathcal{I}. \end{aligned}$$

By (1.22), this gives a representation of the solution to the storage valuation problem (1.6):

$$v(0,s,c,i) = \sup_{u\in\mathcal{U}_0^i} v(0,s,c,i;u) = Y_0.$$

Remark 1.5.1 (Link between optimal switching and impulse control)**.** Optimal switching can be viewed as a particular case of impulse control in higher dimension, namely when adding to the system a pure jump process, standing for the current operating regime. The optimal switching problem:

$$v(t,x,i) = \sup_{u\in\mathcal{U}_t^i} \mathbb{E}^{(t,x,i)}\left[\int_t^T f(X_r^u,u_r)dr - \sum_{k\geq 1, t<\tau_k\leq T} \kappa(u_{\tau_k^-},u_{\tau_k}) + g(X_T^u,u_T)\right], \tag{1.25}$$

in which the switching strategy $u \equiv (\tau_k,\xi_k)_{k\geq 1}$ takes a form like (1.1), X^u follows the dynamics:

$$X_s^u = X_t^u + \int_t^s b(X_r^u,u_r)dr + \int_t^s \Sigma(X_r^u)dW_r, \quad \forall s\geq t,$$

and $\mathbb{E}^{(t,x,i)}[\cdot] := \mathbb{E}[\cdot|X_t^u = x, \tau_0 = t, \xi_0 = i]$ can be rewritten explicitly as an *impulse control problem* as follows. Introduce a pure jump controlled process I^u giving the operating mode at any considered time. The *impulse control* is the sequence $u = (\tau_k,\xi_k)_{k\geq 1}$, which makes the state variable I^u jump from one mode to another: I^u is a $\mathcal{I}$-valued variable constant on each $[\tau_k,\tau_{k+1})$, which jumps to mode ξ_k at impulse date τ_k. Let us then consider the controlled state variable with jump $\bar{X}^u$:

$$\bar{X}^u = \begin{pmatrix} X \\ I^u \end{pmatrix},$$

in which

$$\begin{aligned} X_s &= X_t + \int_t^s b(X_r,I_r^u)dr + \int_t^s \Sigma(X_r)dW_r, \quad \forall s\geq t. \\ I_s^u &= \sum_{k\geq 0} \xi_k \mathbb{1}_{[\tau_k,\tau_{k+1})}(s) \end{aligned}$$

Then the problem in (1.25) is equivalent to the following impulse control problem:

$$v(t,\bar{x}=(x,i)) = \sup_{u\in\mathcal{U}_t} \mathbb{E}^{(t,\bar{x})}\left[\int_t^T f(\bar{X}_r^u)dr - \sum_{k\geq 1, t<\tau_k\leq T} \kappa(I^u_{\tau_k^-},\xi_k) + g(\bar{X}_T^u)\right], \tag{1.26}$$

where $\mathbb{E}^{(t,\bar{x})}[\cdot] := \mathbb{E}[\cdot|\bar{X}_t^u = \bar{x}]$ and $\mathcal{U}_t$ is the set of impulse controls $u = (\tau_k,\xi_k)_{k\geq 1}$ starting at time t ($\tau_0 = t$): $(\tau_k)_{k\geq 1}$ is an increasing sequence of $\mathbb{F}$-stopping times and ξ_k a $\mathcal{F}_{\tau_k}$-measurable variable valued in $\{-1;0;+1\}$.

1.5.3 Solving BSDEs with constrained jumps and link to Part II

Numerical resolution of BSDEs with constrained jumps The numerical resolution of a BSDE with constrained jumps, as for example (1.24), is a challenging problem. The main difficulty comes from the constraint, which concerns the jump component of the solution $(V(j))_{j\in\mathcal{I}}$. Classical approaches by *projected* schemes (discretely reflected backward schemes) are irrelevant, since the process K, allowing to fulfill the constraint on jumps does not a priori satisfy any

Skorohod type minimality condition.

These kind of BSDEs might be numerically approximated by using a *penalization procedure* and classical backward discrete-time schemes for BSDEs with jumps, see Bouchard and Elie [19]. The constraint on the jump component V is introduced in the driver of the BSDE and penalized with a parameter $p > 0$. The solution to the penalized BSDE with jumps is known to converge to the minimal solution of the BSDE with constrained jumps, see [71] and [49]. However, no convergence rate is available for such an algorithm (in particular, the convergence rate of the error due to penalization).

The numerical resolution of BSDEs with constrained jumps raises various questions:

- The theoretical representation of the exact solution using BSDE with constrained jumps holds for a Poisson measure μ with *any intensity measure* $\lambda > 0$. From a numerical viewpoint, this is apparently not the case and it has to be chosen carefully, at least with respect to the time step of the discrete-time resolution grid. Even for BSDEs with jumps (without constraint), Elie [48] alludes to the critical role of λ in practical applications.

- As already mentioned, no convergence rate is available for the numerical scheme using penalization described above. On one hand, in the literature of BSDEs with constraint(s), only Hamadène and Jeanblanc [61] provide a convergence rate of the error introduced when using a penalization procedure: for a BSDE with two reflecting barriers with uncontrolled forward diffusion, the authors obtain a bound on the error between the exact and the penalized solution of order p^{-1}.

  On the other hand, the impact of the penalization parameter p is well-known in practice: as p increases at fixed discrete-time step, the driver explodes leading to numerical instabilities, see e.g., the numerical experiments of Lemor [80] for the resolution by penalization of a BSDE with one reflecting barrier. However, to our best knowledge, there exists no explicit computation of the discretization error with respect to p in the literature.

- The solving procedure described above does not provide any representation of the optimal impulse/switching strategy during the backward in time recursion. An additional numerical procedure, to be determined, is required.

In Part II of this dissertation, we propose a numerical method for solving a class of impulse control problems based on their representations as BSDEs with constrained jumps. We use an approach by penalization and study the convergence of the numerical method, with respect to the main approximation parameters: the jump intensity λ, the penalization parameter p and the time step. The discrete-time algorithm introduced allows a numerical implementation by using purely simulation-based methods and we perform numerical experiments on some practical cases of impulse control.

Chapter 2

Gas Swing contracts

2.1 General description

Another kind of contract which has been commonplace in the natural gas industry for many years are so-called Take-or-Pay, variable base-load or Swing contracts. These contracts have been designed to allow flexibility of delivery with respect to time decisions and the amount of gas used. With the transition to a deregulated market, they have now to be valuated according to their financial risks.

These contracts gives to their buyer the right to receive greater or smaller amounts of gas at certain dates in the future at a contractual price. Distribution companies for example enter into such contracts to hedge themselves against gas price fluctuations. On the other hand, gas producers selling these contracts hedge themselves by introducing a limited flexibility in the quantity of purchased gas and imposing global purchasing constraints.

More generally, the denomination Swing contracts holds for a more general framework, where the purchased commodity is not necessarily natural gas: it applies to electricity, oil, and any other relatively liquid commodity markets. Even if we include in our review the general literature for commodity-based Swing options valuation, we restrict ourselves to the gas Swing contracts specificities for the following presentation.

Such a gas contract can be decomposed in two parts: a future part and a Swing part. The future contract guarantees that the option seller has to deliver a minimal amount of a gas (base load) to the option buyer at certain times. The Swing part gives the option buyer the right to purchase extra quantities of gas, up to a global (annual) limit. He can either "swing up" or "swing down" the volume of purchase gas, hence the name Swing contract.

Since base load agreements can be easily valuated (does not include optionality), the main challenge consists in the Swing part valuation. In addition to various clauses including constraints (among which, a maximal and minimal constraint on the global volume of purchased gas), additional penalties or rights, the contract seller specifies the so-called *contract prices*, corresponding in financial terms to the strike prices at which the amount of gas can be purchased by the option buyer. Typically, this strike price is indexed on moving averages of various oil prices: for example averages over the last 6 months of gas oil and fuel oil prices. This specific characteristic has been less highlighted in the literature, and to our knowledge never explored from a numerical viewpoint. This raises in particular a challenging difficulty due to high dimensionality. In practice, the strike price is commonly assumed to be deterministic and there is thus potentially a significant value loss due to this approximation.

2.2 Classical formulation of the problem and practical analysis

2.2.1 Formulation of the valuation problem

We consider the practical and classical point of view of a supplying contract which provide to its holder the right to purchase periodically (usually daily) an amount of gas. It means that the possible exercise times are *pre-defined* (and deterministic). These Swing options are different and might not be confused with multiple exercises American options (in continuous time) as considered for example by Carmona and Touzi [27]. In present formulation, we highlight in particular volumetric local and global constraints involved in gas supplying contracts: the flexibility is not reduced to time decisions, but also has to take into account volumes management.

In addition, many other clauses than volume constraints (other constraints, penalties and additional rights) might be involved in gas supplying contracts, providing more or less flexibility to the contract holder, see Remark 2.2.1. In the following, we restrict ourselves to basic volume constraints but on the other side, emphasize on the structure of the Swing contract strike price. Indeed, this implies a path dependance of cashflows which complicates much more the valuation.

Indexed strike price of the contract Let us denote by $S^g = (S^g_t)_{t\geq 0}$ the gas spot price (see Paragraph 1.4 for more details on classically-used gas prices) which is a Markov process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The holder of a supplying contract purchases quantities of gas at some *unitary strike price* $\bar{X}$. $\bar{X}$ is commonly an index based on various commodities (gas, oil, coal, etc) and can be written in the general form

$$\bar{X}_t = K + \sum_{i=1}^{d} \alpha_i \bar{X}^i_t, \tag{2.1}$$

where:

- K is a fixed cost, standing for the fixed part of delivery.
- $S^i, \forall i = 1, \ldots, d$ are correlated commodity prices: gas oil, fuel oil, coal, etc.
- α_i is the weight attributed to commodity i (scaling and weighting parameter).
- for each commodity i involved in the index, $\bar{X}^i$ is the average of its price S^i over the δ months preceding the l last months before the last updating date. Besides, each $\bar{X}^i$ is updated every q months. δ is commonly called the *averaging window*, l, the *time lag* or delay and q the *validity period*. See Figure 2.1.
  That is in mathematical words

$$\bar{X}^i_t = X^i_{\varphi_q(t)}, \quad X^i_t = \int_{t-l-\delta}^{t-l} S^i_r dr, \quad \forall t \geq \delta + l, \tag{2.2}$$

  in which $\varphi_q(t) := q\,\lfloor t/q \rfloor$ corresponds to the last strike updating time before time t (included), that is less or equal than s.
  Typical values for the characteristic triple (δ, l, q) are $(6, 0, 1)$, $(3, 0, 1)$ $(3, 1, 1)$, etc.

We shall suppose that the prices $S^i = (S^i_t)_{t\geq 0}, \forall i = 1, \ldots, d$ are Markov processes defined on $(\Omega, \mathcal{F}, \mathbb{P})$, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$ the natural filtration generated by $\left\{S^g, S^1, \ldots, S^d\right\}$. Each process X^i follows the dynamics

$$dX^i_t = \tfrac{1}{\delta}\left(S^i_{t-l} - S^i_{t-l-\delta}\right) dt, \quad \forall t \geq \delta + l.$$

This shows in particular that even if S^i is Markovian, the process (S^i, X^i) is not.

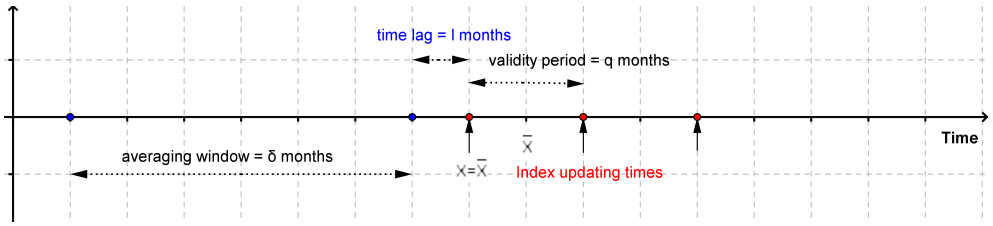

Figure 2.1: 3 characteristic numbers (δ, l, q) of indexed strike prices

Supplying contract value Gas can be purchased at some fixed dates (every day)

$$\{t_0, \ldots, t_{N-1}\}$$

before an expiration date $T = t_N$ (usually, one or two years, from October to October). The unitary purchase price is the above described index $\bar{X}$. In addition, the quantity of purchased gas q_t is subjected to lower and upper local constraints, called minimal and maximal Daily Contract Quantities: $q_{\min} \leq q_t \leq q_{\max}$. Besides, the contract holder is submitted to penalties if he has taken out more or less than specified volumes at the end of the contract. This volumes are called maximal and minimal (Annual) Contract Quantities and will be denoted by $Q_{\max}$ and $Q_{\min}$ respectively.
Let us introduce the value of the indexed strike price at any (discrete) exercise date $t_n \geq \delta + l$. According to common practice, the index is simply computed by using arithmetic averages of the commodity prices $(S^1, S^1, \ldots, S^d)$. We shall consider thus the following discrete-time counterpart of (2.2):

$$\bar{X}^i_{t_n} = X^i_{\varphi_q(t_n)}, \quad X^i_{t_n} = \frac{1}{N_\delta} \sum_{j=n-N_l-N_\delta+1}^{n-N_l} S^i_{t_j}, \quad \forall t_n \geq \delta + l, \tag{2.3}$$

in which N_δ and N_l denote the number of exercises dates within the averaging window with length δ and the time lag respectively. To alleviate notations in the following presentation, we assume that q is equal to one day, meaning that the strike price is updated every day (mathematically: $\varphi_q(t_n) = t_n$). The extension to the general case is straightforward.

Given the dynamics of the various involved commodity prices $(S^g, S^1, S^1, \ldots, S^d)$, the problem is to evaluate the price of the above supplying contract and provide the optimal purchase policy. At time $t = t_n \geq \delta + l$, consider an initial gas price $S^g_t = s$, an initial volume $Q_t = \varrho$ and an initial vector $\vec{x} \in \mathbb{R}^{d\times(N_l+N_\delta)}$ standing for the values of

$$\text{if } t = t_n, \quad X_t := \prod_{i=1}^{d} \left(S^i_{t_n}, S^i_{t_{n-1}}, \ldots, S^i_{t_{n-N_l}}, \ldots, S^i_{t_{n-N_l-N_\delta+1}}\right). \tag{2.4}$$

Then, the contract no-arbitrage value, denoted by $v(t,s,\vec{x},\varrho)$, is solution to the *discrete-time stochastic control problem with constraints*:

$$v(t,s,\vec{x},\varrho) = \sup_{q\in\mathcal{Q}_t} \mathbb{E}\left[\sum_{k=n}^{N-1} q_{t_k}\left(S^g_{t_k} - \bar{X}_{t_k}\right) + g(Q_T)\Big|\mathcal{F}_t\right], \tag{2.5}$$

where:

- the set of admissible controls is

$$\mathcal{Q}_t = \left\{ q=(q_{t_k})_{k=n,\dots,N-1} : \begin{array}{l} q_{t_k} \text{ is a random variable } \mathcal{F}_{t_k}\text{-measurable} \\ \text{such that } q_{\min} \le q_{t_k} \le q_{\max} \end{array} \right\}.$$

- the cumulative volume purchased up to time T (global gas supply) is

$$Q_T = \varrho + \sum_{k=n}^{N-1} q_{t_k}.$$

- the terminal condition imposes a penalty $\beta_{\max} \gg 0$ (resp. $\beta_{\min} \gg 0$) if the cumulative volume purchased until expiracy is above (resp. below) the minimal (resp. maximal) Contract Quantity, namely:

$$g(Q_T) := \mathcal{P}(Q_T, Q_{\min}, Q_{\max}) = -\beta_{\max}\left(Q_T - Q_{\max}\right)^+ - \beta_{\min}\left(Q_{\min} - Q_T\right)^+. \tag{2.6}$$

  Infinite penalties $\beta_{\max} = \beta_{\min} \to +\infty$ (so-called "firm constraints") implies that the contract holder is not allowed to violate the constraints.

- by the Markovian property of the commodity prices $(S^g, S^1, \dots, S^d)$, the definition of the strike price $\bar{X}$ in (2.1)-(2.3) and the definition of X in (2.4), the conditional expectation can be reduced to:

$$\mathbb{E}\left[\cdot|\mathcal{F}_t\right] = \mathbb{E}\left[\cdot|S^g_t = s, X_t = \vec{x}, Q_t = \varrho\right].$$

Remark 2.2.1 (Other clauses involved in supplying contracts)**.** Asche et al. [2] gives a global overview of flexible commodity-based contracts, which may include various clauses. Concerning Swing contracts in the gas market, additional clauses most encountered in practice are Make up and Carry forward rights, rebate and renegotiation clause. For a contract over several years, a make up right allow the buyer to use more flexibility in the current year by reducing the flexibility in the following years. It is the opposite for the carry forward right. A rebate can also be applied if the contract holder has taken out more than a specified volume. Finally, renegotiation usually takes place when the contract price (i.e. the strike $\bar{X}$) is too far from the current market conditions. The coefficients in (2.1) or characteristic triple (δ, l, q) of the various commodities in (2.2) are renegotiated.

2.2.2 Some practical facts for an efficient valuation

Normalization of the contract For both financial and numerical reasons, the contract value is usually split into a *firm contract* and a *normalized Swing option*. The firm contract (base load agreement) can be viewed as a simple set of indexed forward contracts, according which its holder purchases the minimal amount of gas $q_{\min}$ at each exercise date until expiry T (without

any constraint or flexibility). The optional part of the contract corresponds thus to the right of purchasing an amount of gas between 0 and $(q_{\max} - q_{\min})$. By an obvious change of variable, the contract value at time $t = t_n \geq \delta + l$ in (2.5) can be written as

$$v(t, s, \vec{x}, \varrho) = q_{\min} v_{\text{firm}}(t, s, \vec{x}) + (q_{\max} - q_{\min})\, \bar{v}(t, s, \vec{x}, \bar{\varrho})$$

with

$$\begin{aligned} v_{\text{firm}}(t, s, \vec{x}) &= \mathbb{E}\left[\sum_{k=n}^{N-1} \left(S^g_{t_k} - \bar{X}_{t_k}\right) \Big| \mathcal{F}_t\right] \\ \bar{v}(t, s, \vec{x}, \bar{\varrho}) &= \sup_{\bar{q} \in \bar{\mathcal{Q}}_t} \mathbb{E}\left[\sum_{k=n}^{N-1} \bar{q}_{t_k} \left(S^g_{t_k} - \bar{X}_{t_k}\right) + \bar{g}(\bar{Q}_T) \Big| \mathcal{F}_t\right], \end{aligned} \tag{2.7}$$

in which:

- $\bar{\varrho} = \varrho / (q_{\max} - q_{\min})$.
- the set of (normalized) admissible controls is

$$\bar{\mathcal{Q}}_t = \left\{ \begin{array}{ll} \bar{q} = (\bar{q}_{t_k})_{k=n,\ldots,N-1} : & \bar{q}_{t_k} \text{ is a random variable } \mathcal{F}_{t_k}\text{-measurable} \\ & \text{such that } 0 \leq \bar{q}_{t_k} \leq 1 \end{array} \right\}.$$

- the (normalized) cumulative volume purchased up to expiry is $\bar{Q}_T = \bar{\varrho} + \sum_{k=n}^{N-1} \bar{q}_{t_k}$.
- with respect to (2.6), the terminal condition is equal to

$$\bar{g}(\bar{Q}_T) = \mathcal{P}_T\left(\bar{Q}_T, \bar{Q}_{\max} := \frac{Q_{\max} - (N-n) q_{\min}}{q_{\max} - q_{\min}}, \bar{Q}_{\min} := \frac{Q_{\min} - (N-n) q_{\min}}{q_{\max} - q_{\min}}\right).$$

Since the computation of $v_{\text{firm}}(t, s, \vec{x})$ is straightforward, the valuation of the supplying contract boils down to the resolution of (2.7), which constitutes a normalized Swing contract in which the local constraints are $[0, 1]$-valued.

The numerical resolution of problem (2.7) require the use of the well-known dynamic programming principle. The contract value can be computed at initial time $\delta + l$ through the following backward recursion scheme:

$$\begin{aligned} \text{Terminal condition}: \quad & \bar{v}(T, s, \vec{x}, \bar{\varrho}) = \bar{g}(\bar{\varrho}) \\ \text{For } n = N-1, \ldots, N_\delta + N_l: \quad & \bar{v}(t_n, s, \vec{x}, \bar{\varrho}) = \max_{\bar{q} \in [0,1]} \Big\{ \bar{q}(s - \bar{X}_{t_n}) \\ & \qquad + \mathbb{E}^{(t_n, s, \vec{x}, \bar{\varrho})}\left[\bar{v}(t_{n+1}, S^g_{t_{n+1}}, X_{t_{n+1}}, \bar{\varrho} + \bar{q})\right] \Big\} \end{aligned} \tag{2.8}$$

in which $\mathbb{E}^{(t_n, s, \vec{x}, \bar{\varrho})}[\cdot] := \mathbb{E}[\cdot | S^g_{t_n} = s, X_{t_n} = \vec{x}, \bar{Q}_{t_n} = \bar{\varrho}]$ providing also the optimal quantity of gas that the contract holder has to purchase as the arg max of the optimization problem involved in the backward recursion rule. Recall (Markovian framework) that all the information needed to compute $\bar{X}_{t_n}$ is included in vector $\vec{x}$, see (2.4).

Main difficulties This allows us to emphasize on the two main numerical difficulties implied by the resolution of backward recursion scheme in (2.8). First, in the same way as for storage contract valuation, see Paragraph 1.3.2, the cumulative volume Q is policy-dependent. In addition, the controls $\bar{q}$ involved in the optimization problem are continuous-valued. It is thus necessary to discretize first the set of admissible values for $\bar{q}$, that is $\{0, \Delta\bar{q}, \dots, 1 - \Delta\bar{q}, 1\}$, construct a grid of admissible cumulative volumes

$$\Theta = \left\{ \bar{Q}_{\min} + k\Delta\bar{q}, \forall k = 0, \dots, \frac{\bar{Q}_{\max} - \bar{Q}_{\min}}{\Delta\bar{q}} \right\}$$

and then solve the backward scheme (2.8) at each grid point $\bar{\varrho} \in \Theta$. As already mentioned for storage contracts, an accurate numerical resolution using such a discretization technique is sharply memory-demanding. To overcome this first numerical difficulty, the classical approach is to make the following approximation: solve the optimization problem in (2.8) by restricting ourselves to bang-bang strategies, namely to purchase gas quantity valued in $\{0, 1\}$ instead of $[0, 1]$. We refer to the next paragraph for a discussion of such an approximation and resulting (simpler) stochastic control problem.

On the other hand, there is another major computational challenge in dynamic programming equation (2.8) due to *high dimensionality*. Indeed, because of the path-dependence implied by the strike price $\bar{X}$, the dimension of the Markovian state variable, recall (2.4), is equal to

$$d \times (N_\delta + N_l) + 1.$$

This in particular makes it difficult to compute the conditional expectations involved in the backward recursion. For typical gas supplying contracts with strike price of type $(6, 0, 1)$ indexed on gas oil and fuel oil, this lead to a dimension ~ 360, which is computationally unfeasible in practice. To our knowledge, this problem has never been studied in the framework of Swing contract valuation and only by few authors in the framework of American options, see Paragraph 2.3.2. Practitioners most often use two approximations:

(A) Motivated by a rough reduction of dimensionality, a non Markovian approximation is most commonly used: the problem is solved by considering the state variable composed of the gas price and the moving average price, that is $(S^g, \bar{X})$. Since $\bar{X}_t$ depends on the entire history of the commodity prices $(S^1, S^2, \dots, S^d)$ between time t and $t - \delta - l$, this approach introduces a bias (the resulting optimal strategy is suboptimal). Even if, to our knowledge, no theoretical result justifies this approximation, it is widely used in practice.

(B) Worst, some practitioners use a valuation assuming the strike price $\bar{X}$ exogenous, namely *deterministic* in the stochastic problem (2.7). It means that $\bar{X}$ is replaced in (2.7) by some $\bar{X}^{\text{obs}}$, computed from the observed forward curves of the prices $(S^1, S^2, \dots, S^d)$. Morally, this is equivalent to assume a zero volatility coefficient of the commodity price processes implied in the strike price. And so, it provides an "intrinsic" value, which in particular do not take into account the correlation between the gas price and the price of the other commodities: this approximation is apparently not relevant in a context, where commodities are strongly correlated.

Using the bang-bang property For an efficient numerical valuation, one can avoid the costly discretization of the set of admissible volumes, by using an iteration on the number of exercise

rights left allowing to satisfy the global constraints. From now, assume that the terminal penalties $\beta_{\max} = \beta_{\min} \to +\infty$ as it is usually done in practice. The cumulative volume $\bar{Q}_T$ is said to be subjected to *firm constraints* and one can use the results from Bardou et al. [8] about the so-called bang-bang structure of the optimal strategy to roughly simplify the resolution.

Bardou et al. [8] provide a necessary condition for the same problem as (2.7) with firm constraints, ensuring that there exists an optimal strategy of a bang-bang type, that is valued in $\{0,1\}$ instead of $[0,1]$. This condition corresponds simply to an integer-valued condition on the global constraints $(\bar{Q}_{\min}, \bar{Q}_{\max})$, which, in practice, might not be satisfied. However, since the solution $\bar{v}$ is concave and piecewise affine (see [8]) with respect to $(\bar{Q}_{\min}, \bar{Q}_{\max})$, this allows to compute efficiently the solution for all possible ($\mathbb{R}^2$-valued) global constraints $(\bar{Q}_{\min}, \bar{Q}_{\max})$.

Consider the problem in (2.7) with initial purchasing volume $Q_t = \bar{\varrho} \in \mathbb{N}$ at time $t = t_n \geq \delta + l$. If the global constraints are such that $(\bar{Q}_{\min}, \bar{Q}_{\max}) \in \mathbb{N}^2$ and $\bar{\varrho} \leq \bar{Q}_{\min} \leq \bar{Q}_{\max} \leq N - n + \bar{\varrho}$, then there exists an bang-bang optimal strategy $\bar{q}^*$ to problem (2.7), that is $\forall k = n, \ldots, N-1, \bar{q}^*_{t_k}$ is $\{0,1\}$-valued. This result helps to simplify the stochastic control problem (2.7). Indeed, set:

$$n_{\min} := \bar{Q}_{\min} - \bar{\varrho} \in \mathbb{N}, \quad n_{\max} := \bar{Q}_{\max} - \bar{\varrho} \in \mathbb{N}$$

Then, the solution to (2.7) can be written as the solution to a simpler stochastic control problem in which the variables linked to volumetric constraints disappear:

$$\bar{v}(t, s, \vec{x}, \bar{\varrho}) := \bar{v}^{(n_{\min}, n_{\max})}(t, s, \vec{x}) = \sup_{u=(\tau_k)_{k\geq 1} \in \mathcal{U}_t^{(n_{\min}, n_{\max})}} \mathbb{E}\left[\sum_{k \geq 1} \left(S^g_{\tau_k} - \bar{X}_{\tau_k} \right)^+ \Big| \mathcal{F}_t \right], \tag{2.9}$$

where the set of admissible controls is reduced to a set of increasing sequences of stopping times:

$$\mathcal{U}_t^{(n_{\min}, n_{\max})} = \left\{ \begin{array}{ll} u = (\tau_k)_{k\geq 1} : & \tau_k \text{ is a } \mathbb{F}\text{-stopping time valued in } \{t_n, t_{n+1}, \ldots, t_{N-1}\} \\ & \tau_k < \tau_{k+1}, \forall k \geq 1 \\ & n_{\min} \leq \sharp\{k \geq 1, \tau_k < T\} \leq n_{\max} \end{array} \right\}.$$

In fact, $\bar{v}^{(n_{\min}, n_{\max})}$ corresponds to a *simple Swing option* (a Bermudan option with multiple exercises) whose number exercise rights is lower bounded by $n_{\min}$ and upper bounded by $n_{\max}$.

An induction scheme for solving (2.9) is available, which is less memory-consuming than previous backward scheme (2.8). Denote by $v^{(j)}$, the solution to the same problem as (2.9) with at maximum j exercise rights but no minimal exercise rights required, that is with strategies belonging to $\mathcal{U}_t^{(0,j)}$. Then, the Bellman optimality principle provides indeed a direct link between $v^{(j)}$ and $v^{(j-1)}$ ($v^{(0)} = 0$). For any $j = 1, \ldots, n_{\max}$,

$$\begin{array}{l} \text{Terminal condition} : v^{(j)}(T, s, \vec{x}) = 0 \\ \text{For } n = N-1, \ldots, N_\delta + N_l : \\ v^{(j)}(t_n, s, \vec{x}) = \max\Big\{ (s - \bar{X}_{t_n})^+ + \mathbb{E}^{(t_n, s, \vec{x})}\left[v^{(j-1)}(t_{n+1}, S^g_{t_{n+1}}, X_{t_{n+1}}) \right]; \\ \qquad\qquad \mathbb{E}^{(t_n, s, \vec{x})}\left[v^{(j)}(t_{n+1}, S^g_{t_{n+1}}, X_{t_{n+1}}) \right] \Big\} \end{array} \tag{2.10}$$

in which $\mathbb{E}^{t_n, s, \vec{x}}[\cdot] := \mathbb{E}\left[\cdot | S^g_{t_n} = s, X_{t_n} = \vec{x} \right]$. This implies finally $\bar{v}^{(n_{\min}, n_{\max})} = v^{(n_{\max})}$ since $\bar{v}$ is clearly increasing in the number of maximal exercise rights.

2.3 Review of existing methods

In the literature of Swing option valuation[1], only a few clauses involved in the realistic supplying contracts are treated or discussed. See Remark 2.2.1 and Løland and Lindqvist [81] for a discussion about the potential impact of such clauses on the contract value. The only ones that are widely included in the modelization are the local and global volumetric constraints. Either terminal penalty conditions are imposed as in (2.6) or firm global constraints are considered. In the literature review that we propose here, the authors consider the Swing option valuation problem as formulated in (2.5), (2.7) or (2.9). Besides, various gas spot price models are considered (including multi-factor models sometimes including spikes), which have a form close to the one previously introduced, see Paragraph 1.4.

However, we have found no literature on Swing option pricing, taking into account the path-dependency and multi-dimensionality involved in the strike price $\bar{X}$, recall (2.1)-(2.2). The strike price is assumed constant or deterministic for numerical experiments. There is an allusion to the possible path-dependency of (stochastic) strike price $\bar{X}$ in Bardou et al. [7, 8], but from a numerical viewpoint, the authors use a constant strike price. Even not in the framework of Swing options, we include, in Paragraph 2.3.2, a short literature review for the related problem of pricing *moving average American style* options[2].

2.3.1 Classical Swing option valuation

Non-simulation based methods

Wilhelm and Winter [112] applies finite element methods for the pricing of Swing options, by transforming the problem into a series of single stopping time problems. Kjaer [72] deals with a gas model with jumps and introduce the bi-dimensional (volume and gas price) parabolic partial integro-differential equation associated to the problem, and solve it by finite differences.

Jaillet et al. [67] propose a valuation method using on a forest of recombining trinomial trees for Swing contracts with a variable volume per exercise. Considering a one factor gas price model, and using a discretization of the usage amount (say L steps), a multi-layer tree (each layer corresponds to a number of remaining exercise rights) is constructed and the backward induction scheme is solved in three dimensions (price, number of exercise rights left and volume). The number of trees necessary to value a Swing option with n exercise rights is thus of order $L \times n^2$. A similar approach using binomial trees is used by Lari-Lavassani et al. [77] and the authors deal both with a one-factor and two-factor model. Kluge [73] use a Gaussian quadrature method and deal with a two-factor model including spikes. All the above methods are subjected to curse of dimensionality and in particular could not handle with (even not path-dependent) multi-dimensional strike prices indexed on various oil prices, as it is the case in practice.

Bardou et al. [7, 8] propose a quantization approach for estimating the conditional expectations involved in the backward recursion rule. This approach, less limited with respect to the dimension of the underlying, is said to be faster and more precise than the simulation-based least squares Monte Carlo method.

[1] The review in Section 2.3 includes the literature of electricity-based Swing option valuation as well.
[2] A more detailed literature review is presented in Chapter 1 of Part III of this dissertation

Simulation based methods

Many authors use the well-known least squares Monte Carlo method (see Longstaff and Schwartz [82]) and extend the LS algorithm for American/Bermudan options to Swing options, that is for multiple exercise rights, with respect to the volume constraints. Meinshausen and Hambly [93], Gravås [58], Dörr [40] and Figueroa [52] use such a valuation methodology and consider various commodity prices model: one-factor/two-factor mean-reverting models sometimes with jumps. Thanawalla [108] proposes also the use of non-parametric regression using splines for approximating conditional expectations. Ibáñez [66] propose another Monte Carlo method, based on the estimation of the optimal exercise frontier.

Other point of view

In a continuous time setting, by introducing a delay between exercise times, Carmona and Touzi [27] address the problem of Swing options from the perspective of multiple optimal stopping problems. In a Black and Scholes framework, the authors use both Monte Carlo and Malliavin calculus based simulation methods for computing the conditional expectations involved in the backward recursive rule. This approach is extended to Lévy processes by Zeghal and Mnif [116].

Dahlgren [39] proposes a continuous-time model for pricing commodity-based Swing options, incorporating a recovery time during which the option holder cannot exercise. This recovery time can depend on the last amount of commodity purchased. The resulting system of discrete variational inequalities is solved by a finite elements scheme. In the same framework, Zhang and Oosterlee [95] proposes alternative numerical methods based on Fourier cosine expansions and handle with prices model including mean-reversion and jumps.

2.3.2 Valuation of moving average options and link to Part III

Valuation of moving average options In the literature, very few articles discuss moving average options with early exercise feature: Bilger [15], Grau [57], Broadie and Cao [22], Kao and Lyuu [68] and Dai and Al. [38]. We refer the reader to the introductive Chapter 1 of Part III of this thesis for a more detailed review of this research field.

As already mentioned, see (A), a common approach (see e.g., Broadie and Cao [22]) is to use a non Markovian approximate method which consists in computing the conditional expectations implied in the backward induction algorithm using only two explanatory variables: the underlying price value at the considered time and its moving average. Bilger [15] heuristically improves this approach by considering a state vector composed of the underlying price, its moving average and additional partial averages of the price over the rolling period.

To our best knowledge, other numerical approaches are computationally limited to applications where the moving average window is small. In a discrete-time setting, Grau [57] uses a least squares Monte Carlo algorithm and Kao and Lyuu [68] a tree-based method, dealing at the most with averaging window with length $N_\delta = 10$ and $N_\delta = 5$.

Approximation of moving average processes From a different point of view, a more general question related to the problem discussed above is that of the *approximation of moving average processes*. One could approximate the indexed strike price $\bar{X}$, recall (2.1) or more specifically the moving average processes of the various commodities involved in $\bar{X}$, that is each X^i defined in (2.2) by some Markovian process.

We report for example in Figure 2.2 (resp. Figure 2.3) the Day Ahead fuel oil price S^{fo} (resp. gas oil price S^{go})[3] (left) and implied moving average $\bar{X}^{fo}$ (resp. $\bar{X}^{go}$) when averaging the oil prices over the 6 or 3 last months (right). Empirical calibration methods could be certainly used to fit averaging commodity prices.

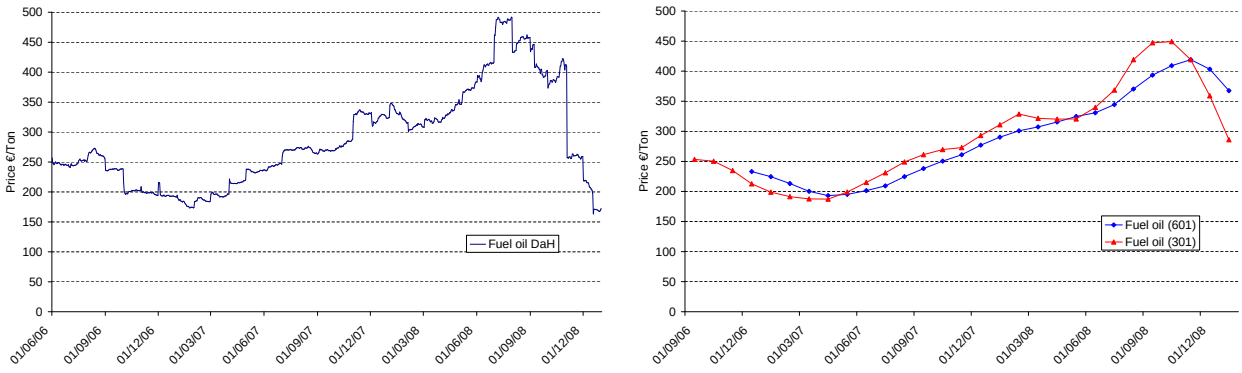

Figure 2.2: Fuel oil DaH prices and corresponding moving averages of type (601) and (301).

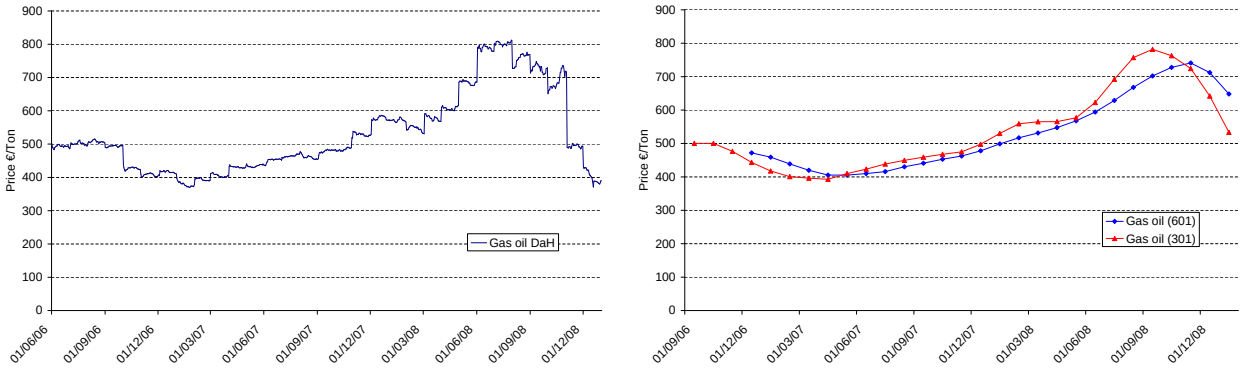

Figure 2.3: Gas oil DaH prices and corresponding moving averages of type (601) and (301).

In Part III of this dissertation, we propose a finite-dimensional approximation of the infinite-dimensional dynamics of moving average processes based on a truncated Laguerre series expansion. With this approximation and in a continuous time setting, the infinite-dimensional problem of moving average option pricing boils down to a finite-dimensional problem. If n denotes the number of terms in the series, the dimension of the resulting problem is equal to $(n + d + 1)$ in the multi-asset framework introduced in Paragraph 2.2.1.

Besides, we introduce a numerical method for solving such an approximate problem based on a least squares Monte Carlo approach. It allows to value moving average options, including large averaging window δ and time lag l and can handle with payoffs involving a strike price with the same form as $\bar{X}$ in (2.1)-(2.2): we perform numerical experiments with realistic oil-indexed strike prices.

[3]Prices from June 2006 to January 2009 observed on the ARA (Amsterdam-Rotterdam-Anvers) oil market for West Europe.

Part II

Solving impulse control problems by using BSDEs with jumps

Chapter 1

Introduction

We consider the following impulse control problem in finite horizon:

$$\sup_{u=(\tau_k)_{k\geq 1}} \mathbb{E}\left[g(X_T^u) + \int_0^T f(X_s^u)ds + \sum_{\substack{k\geq 1 \\ \tau_k \leq T}} \kappa(X^u_{\tau_k^-}) \right].$$

Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$, an impulse control $u = (\tau_k)_{k\geq 1}$ is an increasing sequence of $\mathbb{F}$-stopping times and the controlled state variable X^u is a càdlàg process satisfying SDE

$$X_t^u = X_0^u + \int_0^t b(X_s^u)ds + \int_0^t \sigma(X_s^u)dW_s + \sum_{\tau_k \leq t} \gamma(X^u_{\tau_k^-}), \quad \forall t \geq 0, \tag{1.1}$$

where W is a d-dimensional $\mathbb{F}$-Browning motion. Between two successive intervention times τ_k and τ_{k+1}, the underlying evolves as a diffusion process and the controller makes an integral profit f. At each decided intervention time τ_k, he gives an impulse to the system: the underlying process jumps with a size $\Delta X^u_{\tau_k} = X^u_{\tau_k} - X^u_{\tau_k^-} = \gamma(\tau_k, X^u_{\tau_k^-})$ and he obtains the intervention gain κ.

Let v, be the value function of this impulse control problem, defined by

$$v(t,x) = \sup_{u=(\tau_k)_{k\geq 1} \in \mathcal{U}_{(t,T]}} \mathbb{E}\left[g(X_T^{t,x,u}) + \int_t^T f(X_s^{t,x,u})ds + \sum_{\substack{k\geq 1 \\ t<\tau_k \leq T}} \kappa(X^{t,x,u}_{\tau_k^-}) \right] \tag{1.2}$$

where $\mathcal{U}_{(t,T]}$ is the set of admissible impulse controls valued in $(t,T]$ and $X^{t,x,u}$, the jump diffusion with dynamics (1.1) starting at x in t: under appropriate assumptions, it is well-known (see for example Øksendal and Sulem [94] or more recently Seydel [104]), that v is solution to the quasi-variational inequality (QVI for short):

$$\begin{array}{rcll} \min\left\{ -\frac{\partial v}{\partial t} - \mathcal{L}v - f; v - \mathcal{H}v \right\} & = & 0 & \text{on } [0,T) \times \mathbb{R}^d \\ v(T,\cdot) & = & g & \text{on } \mathbb{R}^d \end{array} \tag{1.3}$$

where $\mathcal{L}$ is the second order local operator of state variable X:

$$\mathcal{L}v(t,x) = b(x) \cdot D_x v(t,x) + \frac{1}{2} \operatorname{tr}\left(\sigma\sigma^{\perp}(x) D_x^2 v(t,x) \right) \tag{1.4}$$

where $\perp$ denotes the matrix transposition and $\mathcal{H}$ is the intervention operator:

$$\mathcal{H}v(t,x) = v(t, x+\gamma(x)) + \kappa(x). \tag{1.5}$$

Equation (1.3) is the dynamic programming equation associated to the impulse control problem (1.2), meaning that at each time before maturity T, the controller may decide whether to do nothing and let the system diffuse or to intervene by giving an impulse.

From a numerical point of view, the main difficulty of the QVI (1.3) lies in that the obstacle term (1.5) contains the solution itself and it is nonlocal. The classical approach for solving impulse control problem (1.2) is based on an iteration on the number of intervention times (see Seydel [104] and Øksendal and Sulem [94] for the infinite horizon case). Indeed, problem (1.2) can be viewed as a *cascade of optimal stopping problems*. Starting from the function

$$u_0(t,x) = \mathbb{E}\left[g(X_T^{t,x,u}) + \int_t^T f(X_s^{t,x,u})ds\right] \tag{1.6}$$

solution to the linear PDE

$$\begin{array}{rcll} -\frac{\partial u_0}{\partial t} - \mathcal{L}u_0 - f &=& 0 & \text{on } [0,T)\times\mathbb{R}^d \\ u_0(T,\cdot) &=& g & \text{on } \mathbb{R}^d, \end{array} \tag{1.7}$$

a sequence of functions $(u_n)_{n\geq 1}$ is constructed by induction and its limit as n goes to infinity gives the solution v to (1.2). At step $n \geq 1$, the solution u_n to the problem with n impulsion times is solution to the optimal stopping problem

$$u_n(t,x) = \sup_{\tau\in\mathcal{T}_{(t,T]}} \mathbb{E}\left[\mathcal{H}u_{n-1}(\ X_\tau^{t,x,u})\right] \tag{1.8}$$

which satisfies the obstacle PDE:

$$\begin{array}{rcll} \min\left\{-\frac{\partial u_n}{\partial t} - \mathcal{L}u_n - f; u_n - \mathcal{H}u_{n-1}\right\} &=& 0 & \text{on } [0,T)\times\mathbb{R}^d \\ u_n(T,\cdot) &=& \mathcal{H}u_{n-1}(T,\cdot) & \text{on } \mathbb{R}^d. \end{array} \tag{1.9}$$

For a numerical computation of the solution, one can use a probabilistic approach when solving (1.6)-(1.8) (see for example Carmona and Touzi [27] in the case of Swing options) or solve the sequence of PDEs (1.7)-(1.9) (see Chancelier et al. [28] and Seydel [104]). Both these numerical methods are computationally demanding since they require at each induction step $n \geq 1$ the resolution of an optimal stopping problem. In particular, the computational time increases with the number of optimal intervention times. In addition, as the obstacle involves a non-local term (cf. (1.5)) at step n, one needs to keep in memory the values of function u_{n-1} on the whole state domain.

An alternative method to solve problem (1.2) is to use the representation of its solution as the solution to a BSDE with constrained jumps. We refer to Kharroubi et al. [71] for a detailed description of this approach. Namely, under appropriate assumptions, the minimal solution (Y,Z,V,K) to

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s - \int_t^T V_s dN_s + \int_t^T dK_s,\ \forall 0\leq t\leq T \\ -V_t \geq \kappa(X_{t^-}),\ \forall 0\leq t\leq T \end{cases} \tag{1.10}$$

in which N is a Poisson process with jump intensity $\lambda > 0$ and X is the (uncontrolled) jump diffusion process:

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \gamma(X_{s^-})dN_s, \quad \forall t \geq 0 \tag{1.11}$$

gives the solution $Y_t = v(t, X_t)$ to initial problem (1.2), whatever the value of the jump intensity λ. $(Y_t)_{t\geq 0}$ is called the value process and jumps with a size $V_t = Y_t - Y_{t^-} = v(t, X_{t^-} + \gamma(X_{t^-})) - v(t, X_{t^-})$. The process K is a non-decreasing process which allows to fulfill the constraint on the jump component V and the solution (Y, Z, V, K) is said to be *minimal* if and only if it has the smallest component Y in the (infinite) class of solutions to (1.10).

We should point out that our problem is related to that of optimal switching. Optimal switching can be viewed as a particular case of impulse control in higher dimension, namely when adding to the system a pure jump process (standing for the current operating regime) controlled by the regime decision, see Remark 1.5.1. We refer for example to Chancelier et al. [28], Maroso [92], Djehiche et al. [45] and Bouchard [17] for more detailed description of this problem in different mathematical frameworks. On the contrary to the general impulse control case, an equivalent formulation of this problem with a *local intervention* (or switching) operator is available (see (1.8) for the specific problem of storage valuation introduced in Chapter 1 in Part I). In Elie and Kharroubi [49], following the approach developed by Kharroubi et al. [71], optimal switching problems with controlled forward processes are related to BSDEs with constrained jumps. However, the solution to this kind of switching problem is also characterized as the solution to a class of multi-dimensional BSDE with oblique reflection, as previously introduced by Hu and Tang [65] and Djehiche et al. [45].

The numerical resolution of BSDEs with constrained jumps, such as (1.10), is a challenging problem. A discrete-time backward scheme for solving BSDEs with jumps is introduced by Bouchard and Elie [19]. In such a case (without constraint), under relevant assumptions on the FBSDE coefficients, the authors obtain a convergence rate of order $|\pi|^{\frac{1}{2}}$ when the time step is equal to $|\pi|$. Besides, to our best knowledge, only Elie [48] has published numerical experiments involving BSDEs with jumps. In particular, the author alludes to the critical role of the jump intensity λ, whereas from a theoretical viewpoint this parameter might be chosen arbitrary.

The difficulties in our case come from the constraint in (1.10), which concerns the jumps component of the solution, namely V and the characterization of the solution to problem (1.2) as the *minimal* solution to constrained BSDE (1.10). This last feature of the solution seems untractable for numerical issues.

Reflected BSDEs, in which the component Y is forced to stay above a given obstacle (reflection of *normal* type) can be solved by using so-called discretely reflected backward schemes, which in particular rely on the Skorohod minimality condition satisfied by the process K in such cases. These schemes have been studied among others by Ma and Zhang [87] and Bouchard and Chassagneux [18]. Under standard assumptions on the FBSDE coefficients, these latter authors obtain a $|\pi|^{\frac{1}{2}}$ convergence rate. This kind of approach by *projection of the contraint* can be used in the framework of optimal switching problems as well, by using the representation of the solution as the solution to the associated multi-dimensional BSDE with oblique reflection. Recently, Chassagneux et al. [32] have introduced such a discretely obliquely reflected numerical scheme and obtain an convergence rate of order $|\pi|^{\frac{1}{2}-\epsilon}$ for any $\epsilon > 0$. However, this result holds

in a framework where the forward process is *uncontrolled*.

Only few authors have published numerical experiments involving (reflected) BSDEs related to optimal switching problems [99, 61, 83]. Porchet [99] uses a numerical scheme by projection for the 2-dimensional reflected BSDE associated to a real option problem (valuation by utility indifference of a power plant with two modes). The case with 2 regimes simplifies the computation of the solution as the implied two-dimensional reflected BSDE can be reduced to a single BSDE with two reflecting barriers (by working on the difference value process). Hamadène and Jeanblanc [61], dealing with a 2-regimes optimal switching problem (starting-stopping problem), use a penalization procedure for such a doubly reflected BSDE. In a specific framework (in particular: uncontrolled forward diffusion and constant switching costs), the authors obtain a convergence rate between the exact and the penalized solution of order p^{-1}.

Finally, Ludkovski [83] solves a 3-regimes switching problem by considering a cascade of RBDSEs with one reflecting barrier (using an iteration on the number of switches), see also Carmona and Ludkovski [25]. By numerical observations, the authors conjecture that the global error of their numerical method grows linearly in the maximum number of switches.

An approach by projection is irrelevant in our case, since the process K, allowing to fulfill the constraint on the jump component V, does not a priori satisfy any Skorohod type minimality condition. We thus consider an approach for solving BSDE with constrained jumps (1.10), based on the *penalization of the obstacle constraint* [1]. More precisely, the constraint on jumps $-V_t \geq \kappa(t, X_{t^-})$ is introduced in the driver of the BSDE and penalized with a parameter $p > 0$. In particular, the term K for which we do not have any tractable interpretation disappears. This leads to a simpler BSDE with jumps

$$Y_t^p = g(X_T) + \int_t^T \left[f(X_s) + p \left(V_s^p + \kappa(X_{s^-}) \right)^+ \lambda \right] ds - \int_t^T Z_s^p dW_s - \int_t^T V_s^p dN_s, \qquad (1.12)$$

whose unique solution (Y^p, Z^p, V^p) tends to the minimal solution (Y, Z, V) to (1.10) as $p \to +\infty$, see [71]. In a second step, we consider a backward discrete-time scheme for numerically solving the penalized BSDE with jumps (1.10) as the one introduced by Bouchard and Elie [19].

Our main contributions are the following: we provide a global convergence rate of the error introduced by the numerical approximation by penalization described above, with respect to the penalization parameter p, the jump intensity λ and the time step $|\pi|$. We perform numerical experiments on some practical cases of impulse control in the framework of real options. The results that we obtain allow a better understanding of the impact of the jump intensity and the penalization parameter from a numerical viewpoint.

Firstly, we derive an explicit rate of convergence of the error due to penalization. Denote by $(Y^{p,t,x}, Z^{p,t,x}, V^{p,t,x})$ the solution to (1.12) when $X \equiv (X_s^{t,x})_{t \leq s \leq T}$ is the solution starting at x in t to SDE (1.11). We use an explicit functional representation for $Y^{p,t,x}$ as an essential supremum over a family of probability measures which only impact the jump intensity of N. By a convenient change of measure, namely a measure change which forces the penalized solution to jump as soon as possible after that an optimal impulse occurs, together with a continuity argument of the value function in its maturity variable, we prove that:

$$\sup_{0 \leq t \leq T} \left| v(t,x) - Y_t^{p,t,x} \right| \leq C(\lambda p)^{\alpha - \frac{1}{2}}, \quad \forall \alpha \in \left(0, \frac{1}{2}\right).$$

[1] This penalization approach can be viewed as an alternative approach to iteration. This allows us to avoid classical inductive schemes based on an iteration on the number of interventions.

Secondly, we carefully perform an estimation of the discretization error introduced by our numerical scheme in terms of additional parameters λ and p, by using the same arguments as Bouchard and Elie [19]. We prove that a necessary condition for a convergence of order $|\pi|^{\frac{1}{2}}$ of the backward discrete-time scheme is

$$|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right).$$

Besides, the discretization error is shown to exponentially grow with (λp^2).

More generally, the numerical method that we consider allows to handle a relatively large class of stochastic control problems commonly encountered in the real options framework. This includes in particular impulse control problem with *controlled* forward processes with possibly *degenerate* dynamics (forward processes with a zero diffusion term) and it is also applicable to optimal switching problems.

The rest of this part is structured as follows: in Chapter 2, we set the considered impulse control problem in the mathematical framework of BSDEs with constrained jumps. We also provide a property of the value function (1.2), which, to our knowledge, is new in the framework of impulse control: it is $\frac{1}{2}$-Hölder with respect to its maturity variable. In Chapter 3 and 4, we respectively deal with the penalization error and the discretization error introduced by our numerical approximation. In Theorem 4.4.1, we provide the global approximation error that we obtain in terms of parameters $(\lambda, p, |\pi|)$.

In Chapter 5, we illustrate our approach with two examples, including numerical experiments. The numerical method is implemented via pure simulation-based techniques. In particular, we use a least squares Monte Carlo approach for estimating the conditional expectations operators involved in the backward induction scheme.

We first deal with a forest optimal management problem proposed by Øksendal and Sulem [94] (see also Willassen [113]). We then apply our approach for the valuation of simple Swing options with $n_{\max} \geq 1$ exercise rights (options with a multiple-exercise feature, see e.g., Carmona and Touzi [27]). This constitutes a particularly degenerate three-dimensional impulse control problem.

Finally, we propose a numerical algorithm for the valuation of gas storage facilities, which is based on the same methodology as the one presented above. This real option problem has been presented in Part I and belongs to the class of optimal switching problem, involving in particular a *degenerate, controlled and constrained* variable (namely, the inventory level). Some adaptations of our method are required to obtain relevant valuation results, and this last challenging task is left for further research.

We report in the final chapter some concluding remarks and directions for further research.

Chapter 2

An impulse control problem: link to BSDEs with jumps

2.1 Notations and assumptions

Throughout this chapter and the two following Chapter 3 and 4, we work in a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, on which is defined a d-dimensional Brownian motion W and a Poisson process N with intensity $\lambda > 0$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, the augmentation of natural filtration generated by W and N, by $\mathbb{F}^W = (\mathcal{F}_t^W)_{t\geq 0}$ the one generated by W, and by $\mathcal{P}$, the σ-algebra of predictable sub-sets of $\Omega \times [0,T]$.

We will denote by

- $\mathcal{S}^2_{[s,r]}$, the set of real-valued càdlàg adapted processes $Y = (Y_t)_{s\leq t\leq r}$ such that

$$\|Y\|_{\mathcal{S}^2_{[s,r]}} = \left(\mathbb{E}\left[\sup_{s\leq t\leq r} |Y_t|^2\right]\right)^{\frac{1}{2}} < \infty$$

  with shorthand notation $\mathcal{S}^2 := \mathcal{S}^2_{[0,T]}$,

- $\mathcal{A}^2$, the sub-set of $\mathcal{S}^2$ such that

$$\mathcal{A}^2 = \left\{K \in \mathcal{S}^2 : (K_t)_{0\leq t\leq T} \text{ nondecreasing }, K_0 = 0\right\},$$

- $\mathrm{L}^p_{\mathbb{F}}([0,T])$, the set of real-valued adapted processes $(\phi_t)_{0\leq t\leq T}$ such that

$$\mathbb{E}\left[\int_0^T |\phi_t|^p \, dt\right] < \infty,$$

- $\mathrm{L}^p_{[s,r]}(W)$, the set of real-valued $\mathcal{P}$-measurable processes $Z = (Z_t)_{s\leq t\leq r}$ such that

$$\|Z\|_{\mathrm{L}^p_{[s,r]}(W)} = \left(\mathbb{E}\left[\int_s^r |Z_t|^p \, dt\right]\right)^{\frac{1}{p}} < \infty$$

  with shorthand notation $\mathrm{L}^p(W) := \mathrm{L}^p_{[0,T]}(W)$,

- $\mathrm{L}^p_{[s,r]}(N)$, the set of real-valued $\mathcal{P}$-measurable processes $U = (U_t)_{s\leq t\leq r}$ such that

$$\|U\|_{\mathrm{L}^p_{[s,r]}(N)} = \left(\mathbb{E}\left[\int_s^r |U_t|^p \, \lambda dt\right]\right)^{\frac{1}{p}} < \infty$$

  with shorthand notation $\mathrm{L}^p(N) := \mathrm{L}^p_{[0,T]}(N)$.

In the following, we will consider assumptions:

(H) (H_X) $b : \mathbb{R}^d \mapsto \mathbb{R}^d$, $\sigma : \mathbb{R}^d \mapsto \mathbb{R}^{d\times d}$ and $\gamma : \mathbb{R}^d \mapsto \mathbb{R}^d$ are Lipschitz continuous and γ is uniformly bounded.

(H_Y) $f : \mathbb{R}^d \mapsto \mathbb{R}$, $\kappa : \mathbb{R}^d \mapsto \mathbb{R}$ and $g : \mathbb{R}^d \mapsto \mathbb{R}$ are Lipschitz continous.

(H_1) There exists a solution $(\bar{Y}, \bar{Z}, \bar{K}) \in \mathcal{S}^2 \times \mathrm{L}^2(W) \times \mathcal{A}^2$ to

$$Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s + \int_t^T \kappa(X_{s^-})dN_s + \int_t^T dK_s.$$

(H_1') (H_1) holds and $\bar{Y}_t = \bar{v}(t, X_t), \forall 0 \le t \le T$ for some $\bar{v}$ with linear growth

(H_2) There exists a non negative function $\varphi \in \mathcal{C}^2(\mathbb{R}^d)$ and a positive constant ρ such that

(i) $\mathcal{L}\varphi + f \le \rho\varphi$,

(ii) $\varphi - \mathcal{H}\varphi > 0$,

(iii) $\varphi \ge g$,

(iv) $\lim_{|x|\to\infty} \frac{\varphi(x)}{1+|x|} = \infty$.

An impulse strategy $u = (\tau_k)_{k\ge1}$ is said to be admissible for problem (1.2) (and belongs to $\mathcal{U}_{(t,T]}$) if $(\tau_k)_{k\ge1}$ is an increasing sequence of $\mathbb{F}^W$-stopping times valued in $(t, T]$ (we set by convention $\tau_0 = t$) such that, if

$$n^u_{(t,T]} := \sharp\{k \ge 1 : t < \tau_k \le T\} \tag{2.1}$$

denotes the number of impulses in strategy u,

$$\mathbb{E}\left|n^u_{(t,T]}\right|^2 < C, \tag{2.2}$$

for some universal constant C. For a strategy $u \in \mathcal{U}_{(t,T]}$, we will denote by $X^{t,x,u}$ the solution to (1.1) starting in x at time t. When assuming (H_X), a straightforward computation using Gronwall's lemma and (2.2) shows that

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad \mathbb{E}\left[\sup_{t\le s\le T}\left|X^{t,x,u}_s\right|^2\right] < C, \tag{2.3}$$

for another constant C.

Troughout this part, we will suppose as prerequired the existence of an optimal strategy $u^* = (\tau^*_k)_{k\ge1} \in \mathcal{U}_{(t,T]}$ to problem (1.2). We refer for example to Bensoussan and Lions [13] and Øksendal and Sulem [94] in the infinite horizon case, for specific conditions on the coefficients of the problem which ensures such an existence of a solution.

Remark 2.1.1. In a specific impulse control framework, Djehiche et al. [44] show the existence of an optimal strategy. So do El-Asri and Hamadène [47] in a multiple optimal switching framework (with an uncontrolled forward diffusion). In such a framework, Chassagneux et al. [32] show in addition that the optimal number of switches belongs to L^2.

Remark 2.1.2. Assumption (H) implies in particular that f, κ, g b, σ and γ satisfy a linear growth condition, namely:

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|\psi(t, x)|}{1 + |x|} < \infty. \tag{2.4}$$

We shall assume finally as Øksendal and Sulem [94] and Seydel [104] that the value function v defined in (1.2) satisfies the linear growth condition (2.4).

Notation Throughout this part, the euclidean norm defined on $\mathbb{R}^d$ or on $\mathbb{R}$ will be indiscriminately denoted by $|\cdot|$. In addition, unless specified otherwise, C will denote a strictly positive constant (which may change from line to line) depending only on Lipschitz coefficients of b, σ, γ, f, κ and g and the bound for γ in (H) and constants T, $|b(0)|$, $|\sigma(0)|$, $|\gamma(0)|$, $|f(0)|$, $|\kappa(0)|$ and $|g(0)|$.

2.2 Link to BSDE with constrained jumps and penalization approach

In this section, we present the results issued from Kharroubi et al. [71] which link the impulse control problem (1.2) to the Backward SDE with constrained jumps

$$\begin{cases} Y_t = g(X_T) + \int_t^T f(X_s)ds - \int_t^T Z_s dW_s - \int_t^T V_s dN_s + \int_t^T dK_s, \ \forall 0 \le t \le T \\ -V_t \ge \kappa(X_{t^-}), \ \forall 0 \le t \le T \end{cases} \tag{2.5}$$

in which X is the solution to forward SDE with jumps

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \gamma(X_{s^-})dN_s, \quad \forall t \ge 0. \tag{2.6}$$

This SDE admits an unique solution in $\mathcal{S}^2$ under (H_X). Adding assumptions (H_Y) and (H_1), (2.5) admits an unique minimal solution $(Y, Z, U, K) \in \mathcal{S}^2 \times \mathrm{L}^2(W) \times \mathrm{L}^2(N) \times \mathcal{S}^{2,c}$ with K predictable - minimal in the sense that the Y component is smaller than any other solution, see [71].

The proof of existence of a minimal solution to BSDE with constrained jump (2.5) in [71] relies on super-martingale representation theorems and measure change arguments (to be linked with a dual formulation of the associated stochastic target problem). As a by-product it gives the following functional representation for the value process Y (we refer to [17] for these ideas previously developed in the framework of optimal switching problems) under assumptions (H) and (H_1):

$$Y_t = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^\nu \left[g(X_T) + \int_t^T f(s, X_s)ds + \int_t^T \kappa(s, X_s)dN_s | \mathcal{F}_t \right], \quad \forall 0 \le t \le T \tag{2.7}$$

where $\mathbb{E}^\nu$ denotes the expectation operator under $\mathbb{P}^\nu$ defined as follows. $\mathcal{V}$ is the set of $\mathcal{P}$-measurable essentially bounded processes, valued in $(0, \infty)$ such that, for any $\nu \in \mathcal{V}$, the equivalent probability measure $\mathbb{P}^\nu$ to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ has Radon-Nikodym density

$$\left. \frac{d\mathbb{P}^\nu}{d\mathbb{P}} \right|_{\mathcal{F}_T} = e^{-\int_0^T (\nu_s - 1)\lambda ds} e^{\int_0^T \ln(\nu_s) dN_s}. \tag{2.8}$$

The specificity of such a change of measure is that it impacts only the jump parts of the processes: the Brownian motion W remains unchanged whereas N has an intensity $(\lambda\nu_s)_{s \ge 0}$.

Now, from the Markov property of jump diffusion X and uniqueness of minimal solution to (2.5), there exists a deterministic function v such that $Y_t = v(t, X_t), \forall 0 \le t \le T$ where

$$v(t, x) = Y_t^{t,x}, \forall (t, x) \in [0, T] \times \mathbb{R}^d \tag{2.9}$$

in which $(X_s^{t,x})_{t \le s \le T}$ is the solution to (2.6) starting in x at t and $(Y_s^{t,x}, Z_s^{t,x}, V_s^{t,x}, K_s^{t,x})_{t \le s \le T}$ the minimal solution to (2.5) with $(X_s)_{t \le s \le T} = (X_s^{t,x})_{t \le s \le T}$.

Proposition 2.2.1. *Assume* (H) *and* (H_1'). *v defined in* (2.9) *is a viscosity solution to QVI*

$$\begin{array}{rcl} \min\left\{-\frac{\partial v}{\partial t}(t,x)-\mathcal{L}v(t,x)-f(t,x)\ ;\right. & & \\ \left. v(t,x)-\mathcal{H}v(t,x)\right\} & = & 0 \quad \forall t\in[0,T), \forall x\in\mathbb{R}^d \\ \min\left\{v(T^-,x)-g(x); v(T^-,x)-\mathcal{H}v(T^-,x)\right\} & = & 0 \quad \forall x\in\mathbb{R}^d \end{array} \tag{2.10}$$

in which operators $\mathcal{L}$ and $\mathcal{H}$ has been defined in (1.4) *and* (1.5). *Adding condition* (H_2), *v is the unique solution to PDE* (2.10) *which satisfies the linear growth condition* (2.4) *and v is continuous on $[0,T)\times\mathbb{R}^d$.*

In general, the terminal condition $v(T^-,\cdot)=g$ is irrelevant, because of the possible discontinuity of Y in T^- due to constraints: relaxed terminal condition in (2.10) expresses the possibility of a jump at time T^-.

Remark 2.2.1. For a better intuition, the following interpretation to solution (Y,Z,V,K) holds when assuming $v\in\mathcal{C}^{1,2}([0,T],\mathbb{R}^d)$:

$$\begin{array}{rcl} \forall 0\le t\le T, \quad Y_t & = & v(t,X_t) \\ Z_t & = & \sigma(t,X_{t^-})D_x v(t,X_{t^-}) \\ V_t & = & v(t,X_{t^-}+\gamma(t,X_{t^-}))-v(t,X_{t^-}) \\ & = & \mathcal{H}v(t,X_{t^-})-v(t,X_{t^-})-\kappa(t,X_{t^-}) \\ K_t & = & \int_0^t\left(-\frac{\partial v}{\partial t}-\mathcal{L}v-f\right)(s,X_s)ds. \end{array}$$

The constraint in (2.5) means thus that the obstacle condition is satisfied, namely $v(t,X_{t^-})-\mathcal{H}v(t,X_{t^-})\ge 0$.

Now go back to the initial impulse control problem (1.2) whose value function, solution to QVI (1.3), is assumed to satisfy linear growth condition (2.4).

Corollary 2.2.1. *Assume* (H), (H_1') *and* (H_2). *Let $(X_s^{t,x})_{t\le s\le T}$ be the solution to* (2.6) *starting in x at t and $(Y_s^{t,x},Z_s^{t,x},V_s^{t,x},K_s^{t,x})_{t\le s\le T}$ the minimal solution to* (2.5) *with underlying $(X_s)_{t\le s\le T}=(X_s^{t,x})_{t\le s\le T}$. Then the solution to impulse control problem* (1.2) *coincides with initial value of component $Y^{t,x}$:*

$$Y_t^{t,x}=v(t,x)=\sup_{u=(\tau_k)_{k\ge1}\in\mathcal{U}_{(t,T]}}\mathbb{E}\left[\int_t^T f(s,X_s^{t,x,u})ds+\sum_{\substack{k\ge1\\ t<\tau_k\le T}}\kappa(\tau_k,X_{\tau_k^-}^{t,x,u})+g(X_T^{t,x,u})\right] \tag{2.11}$$

as (unique) solution with linear growth to QVI (2.10). *Besides v is continuous on $[0,T)\times\mathbb{R}^d$.*

Together with (2.7), Corollary 2.2.1 means intuitively that in the impulse control problem (1.2), the supremum over all impulse control strategies can be substituted by the supremum over a family of probability measure $(\mathbb{P}^\nu)_{\nu\in\mathcal{V}}$ which only impacts the intensity of the jumps of N.

For a fixed penalization parameter $p>0$, let us consider the following BSDE with jumps, which corresponds to constrained BSDE (2.5) in which the constraint on these jumps, namely V, penalizes the driver with a coefficient p:

$$Y_t^p=g(X_T)+\int_t^T\left[f(X_s)+p\left(V_s^p+\kappa(X_{s^-})\right)^+\lambda\right]ds-\int_t^T Z_s^p dW_s-\int_t^T V_s^p dN_s, \forall 0\le t\le T \tag{2.12}$$

Under (H), this BSDE admits an unique solution $(Y^p, Z^p, U^p) \in \mathcal{S}^2 \times \mathrm{L}^2(W) \times \mathrm{L}^2(N)$ from the classical theory on BSDE with jumps (see for example [9]). We present in the following proposition the existing result of convergence of $(Y^p, Z^p, V^p)_p$ to (Y, Z, V).

Proposition 2.2.2. *Assume* (H) *and* (H_1)*, the solution* $(Y^p, Z^p, V^p)_p$ *to* (2.12) *converges to the minimal solution to* (2.5) *as* $p \to +\infty$ *in the following sense:*

1. $(Y^p)_p$ *converges increasingly to* Y *as* $p \to +\infty$ *and this holds in* $\mathrm{L}^2_{\mathbb{F}}([0,T])$
2. $\|Z^p - Z\|_{\mathrm{L}^2(W)} + \|V^p - V\|_{\mathrm{L}^2(N)} \longrightarrow 0$ *as* $p \to +\infty$
3. K *is the weak limit in* $\mathrm{L}^2_{\mathbb{F}}([0,T])$ *of* $K^p = p\int_0^{\cdot} (V_s^p + \kappa(s, X_{s^-}))^+ \lambda ds$ *as* $p \to +\infty$

In [71], the authors state this convergence result, but do not give any rate of convergence. It does not seem possible to obtain any strong convergence result in $\mathcal{S}^2 \times \mathrm{L}^2(W) \times \mathrm{L}^2(N) \times \mathcal{A}^2$ of the penalized solution (Y^p, Z^p, V^p, K^p) to solution (Y, Z, V, K) to BSDE with constrained jump (2.5) with classical arguments. This is due to the presence of a constraint on component V and the lack of minimality condition for component K in (2.5).

First contribution of our work is to provide a convergence rate of $Y_t^{p,t,x}$ given by the solution to (2.12) when $(X_s)_{t\le s\le T} = (X_s^{t,x})_{t\le s\le T}$, to the exact solution $v(t,x)$ of impulse control problem (1.2), see Chapter 3. Our idea follows from explicit functional representation for Y^p

$$Y_t^p = \operatorname*{ess\,sup}_{\nu^p \in \mathcal{V}^p} \mathbb{E}^{\nu^p}\left[g(X_T) + \int_t^T f(s, X_s) ds + \int_t^T \kappa(s, X_{s^-}) dN_s | \mathcal{F}_t \right], \forall 0 \le t \le T \tag{2.13}$$

where $\mathcal{V}^p = \{\nu^p \in \mathcal{V} | \nu_t^p \le p, \forall 0 \le t \le T\}$ and $\mathbb{E}^{\nu^p}$ denotes the expectation under the probability measure $\mathbb{P}^{\nu^p}$ defined by (2.8). The representation (2.13) can be interpreted as follows: the value function of the considered impulse control problem can be approximated by the value function of the same impulse control problem but restricted to strategies whose numbers of impulses are bounded on average by a constant proportional to λp. Choosing a suitable measure change of this type, we will be able to minimize the distance between the penalized value process and solution to (1.2) as p goes to infinity and obtain a convergence rate with respect to λ and p.

2.3 Hölder property of the value function w.r.t. time maturity

Let us consider the same impulse control problem as (1.2) but with maturity $\theta \le T$ and denote its value function by

$$v_\theta(t,x) = \sup_{u=(\tau_k)_{k\ge 1} \in \mathcal{U}_{(t,\theta]}} \mathbb{E}\left[g(X_\theta^u) + \int_t^\theta f(X_s^u) ds + \sum_{k\ge 1, t<\tau_k\le\theta} \kappa(X_{\tau_k^-}^u) \right].$$

As for initial problem (1.2) for which $v(t,x) = v_T(t,x)$, we assume for any $\theta \in (t,T]$ the existence of an optimal impulse strategy to problem $v_\theta(t,x)$.

In our homogeneous framework, the continuity of v_T on $[0,T) \times \mathbb{R}^d$ (see Corollary 2.2.1) implies the continuity of $\theta \to v_\theta(0,x)$ on $(0,T]$. The following proposition strengthens this regularity property to a $\frac{1}{2}$-Hölder property, by using the functional representation (2.7) for the value process at a time close to maturity T.

Proposition 2.3.1. *Assume* (H), (H_1') *and* (H_2). *Then:*

$$\forall x \in \mathbb{R}^d,\ \theta \to v_\theta(0,x)$$

is $\frac{1}{2}$*-Hölder on* $(0,T]$.

Proof. Let $(T,T') \in \mathbb{R}^2, 0 < T' \leq T$ and $(u^* = (\tau_k^*)_{k\geq 1}, u'^* = (\tau_k'^*)_{k\geq 1}) \in \mathcal{U}_{(0,T]} \times \mathcal{U}_{(0,T']}$ be optimal strategies of problems with maturity T and T' respectively.
Step 1. Extending the definition of u'^* on $\mathcal{U}_{(0,T]}$ by adding an empty set of impulses on $[T',T]$,

$$\begin{aligned} & v_{T'}(0,x) + \mathbb{E}\left[g(X_T^{u'^*}) - g(X_{T'}^{u'^*}) + \int_{T'}^T f(X_s^{u'^*})ds\right] \\ = \ & \mathbb{E}\left[g(X_T^{u'^*}) + \int_0^T f(X_s^{u'^*})ds + \sum_{k\geq 1, \tau_k'^* \leq T} \kappa(X_{\tau_k'^{*-}}^{u'^*})\right] \ \leq \ v_T(0,x) \end{aligned}$$

by sub-optimality of $u'^* \in \mathcal{U}_{(0,T']}$ for v_T. Thus

$$v_{T'}(0,x) - v_T(0,x) \leq -\mathbb{E}\left[g(X_T^{u'^*}) - g(X_{T'}^{u'^*}) + \int_{T'}^T f(X_s^{u'^*})ds\right].$$

By linear growth of f, Lipschitz property of g and Cauchy-Schwartz' inequality,

$$\begin{aligned} \mathbb{E}\left|g(X_T^{u'^*}) - g(X_{T'}^{u'^*}) + \int_{T'}^T f(X_s^{u'^*})ds\right|^2 &\leq 2\mathbb{E}\left|g(X_T^{u'^*}) - g(X_{T'}^{u'^*})\right|^2 + 2\mathbb{E}\left|\int_{T'}^T f(X_s^{u'^*})ds\right|^2 \\ &\leq C\mathbb{E}\left|X_T^{u'^*} - X_{T'}^{u'^*}\right|^2 + C|T-T'|^2\left(1 + \mathbb{E}\left[\sup_{T'\leq s\leq T}|X_s^{u'^*}|^2\right]\right). \end{aligned}$$

By definition of u'^*, $(\hat{X}_s)_{t\geq T'} \equiv (X^{u'^*})_{t\geq T'}$ is an uncontrolled process with dynamics

$$\hat{X}_t = X_{T'}^{u'^*} + \int_{T'}^t b(\hat{X}_t)dt + \int_{T'}^t \sigma(\hat{X}_t)dW_t, \quad \forall t \geq T'.$$

Besides, straightforward computation using (H_X), Doob's inequality, and Gronwall's lemma shows that

$$\begin{aligned} \mathbb{E}\left[\sup_{T'\leq s\leq T}|\hat{X}_s|^2\right] &\leq C\left(1 + \mathbb{E}|X_{T'}^{u'^*}|^2\right), \\ \mathbb{E}\left|\hat{X}_T - \hat{X}_{T'}\right|^2 &\leq C\left(1 + \mathbb{E}|X_{T'}^{u'^*}|^2\right)\left|T-T'\right|. \end{aligned}$$

As $u'^* \in \mathcal{U}_{(0,T']}$, estimate (2.3) gives

$$\mathbb{E}\left|g(X_T^{u'^*}) - g(X_{T'}^{u'^*}) + \int_{T'}^T f(X_s^{u'^*})ds\right|^2 \leq C\left|T-T'\right|,$$

for some constant $C < \infty$ which does not depend either on T or T'. By Jensen's inequality, we get finally

$$v_{T'}(0,x) - v_T(0,x) \leq C\left|T-T'\right|^{\frac{1}{2}}.$$

Step 2. Conversely, dynamic programming principle applied to value function v_T gives

$$v_T(0,x) = \mathbb{E}\left[\int_0^{T'} f(X_s^{u^*})ds + \sum_{k\geq 1, \tau_k^* \leq T'} \kappa(X_{\tau_k^{*-}}^{u^*}) + v_T(T', X_{T'}^{u^*})\right]$$

and

$$\begin{aligned} & v_T(0,x) + \mathbb{E}\left[g(X_{T'}^{u^*}) - v_T(T', X_{T'}^{u^*})\right] \\ = \; & \mathbb{E}\left[g(X_{T'}^{u^*}) + \int_0^{T'} f(X_s^{u^*})ds + \sum_{k\geq 1, \tau_k^* \leq T'} \kappa(X_{\tau_k^{*-}}^{u^*})\right] \;\leq\; v_{T'}(0,x) \end{aligned}$$

by sub-optimality of $u^* \in \mathcal{U}_{(0,T]}$ for $v_{T'}$. We get thus

$$v_T(0,x) - v_{T'}(0,x) \leq -\mathbb{E}\left[g(X_{T'}^{u^*}) - v_T(T', X_{T'}^{u^*})\right].$$

To estimate the right-hand side of previous inequality, we use a functional representation at time T' of the solution to impulse control problem with maturity T. In our Markov setting, we have, see [71], for any $x \in \mathbb{R}^d$:

$$v_T(T',x) = \sup_{\nu\in\mathcal{V}} \underbrace{\mathbb{E}^\nu\left[g(X_T^{T',x}) + \int_{T'}^T f(X_s^{T',x})ds + \int_{T'}^T \kappa(X_{s^-}^{T',x})dN_s\right]}_{:=\, v_T(T',x;\nu)}, \tag{2.14}$$

where $\mathbb{E}^\nu$ denotes the expectation under the probability measure $\mathbb{P}^\nu$ equivalent to $\mathbb{P}$ defined by (2.8) and $(X_t^{T',x})_{t\geq T'}$ satisfies the SDE

$$X_t^{T',x} = x + \int_{T'}^t b(X_s^{T',x})ds + \int_{T'}^t \sigma(X_s^{T',x})dW_s + \int_{T'}^t \gamma(X_s^{T',x})dN_s, \quad \forall t \geq T'. \tag{2.15}$$

From (2.14), there exists some $\bar\nu \in \mathcal{V}$ such that

$$0 \leq v_T(T',x) - v_T(T',x;\bar\nu) \leq |T-T'|^{\frac{1}{2}}, \tag{2.16}$$

Denote by $\bar N$, the Poisson process with intensity $\lambda\bar\nu$ under $\mathbb{P}^{\bar\nu}$ and $\bar X^{T',x}$, the solution to (2.15) driven by $\bar N$. By change of measure,

$$v_T(T',x;\bar\nu) = \mathbb{E}\left[g(\bar X_T^{T',x}) + \int_{T'}^T f(\bar X_s^{T',x})ds + \int_{T'}^T \kappa(\bar X_{s^-}^{T',x})d\bar N_s\right]. \tag{2.17}$$

Then,

$$\begin{aligned} \left|g(x) - v_T(T',x)\right| &\leq \left|g(x) - v_T(T',x;\bar\nu)\right| + \left|v_T(T',x;\bar\nu) - v_T(T',x)\right| \\ &\leq \left|g(x) - v_T(T',x;\bar\nu)\right| + |T-T'|^{\frac{1}{2}} \qquad \text{by (2.16)}. \end{aligned}$$

The first term of the right hand side of previous expression is such that

$$\begin{aligned} \left|g(x) - v_T(T',x;\bar\nu)\right| &= \left|g(x) - \mathbb{E}\left[g(\bar X_T^{T',x}) + \int_{T'}^T f(\bar X_s^{T',x})ds + \int_{T'}^T \kappa(\bar X_{s^-}^{T',x})d\bar N_s\right]\right| \\ &\leq \mathbb{E}\left|g(x) - g\left(\bar X_T^{T',x}\right) - \int_{T'}^T f(\bar X_s^{T',x})ds - \int_{T'}^T \kappa(\bar X_{s^-}^{T',x})d\bar N_s\right| \\ &\leq \mathbb{E}\left|g(x) - g\left(x + \int_{T'}^T b(\bar X_s^{T',x})ds + \int_{T'}^T \sigma(\bar X_s^{T',x})dW_s + \int_{T'}^T \gamma(\bar X_s^{T',x})d\bar N_s\right)\right| \\ &+ \mathbb{E}\left|\int_{T'}^T f(\bar X_s^{T',x})ds\right| + \mathbb{E}\left|\int_{T'}^T \kappa(\bar X_{s^-}^{T',x})d\bar N_s\right| \end{aligned}$$

by definition of $\bar{X}^{T',x}$. By (H), Doob's inequality and boundness of $\bar{\nu}$, we obtain

$$\begin{aligned}
\left|g(x) - v_T(T',x;\bar{\nu})\right|^2 &\leq C\mathbb{E}\left|\int_{T'}^T b(\bar{X}_s^{T',x})ds + \int_{T'}^T \sigma(\bar{X}_s^{T',x})dW_s + \int_{T'}^T \gamma(\bar{X}_s^{T',x})d\bar{N}_s\right|^2 \\
&+ C\mathbb{E}\left|\int_{T'}^T f(\bar{X}_s^{T',x})ds\right|^2 + C\mathbb{E}\left|\int_{T'}^T \kappa(\bar{X}_{s^-}^{T',x})d\bar{N}_s\right|^2 \\
&\leq C|T-T'|\left(1 + \mathbb{E}\left[\sup_{T'\leq s\leq T}\left|\bar{X}_s^{T',x}\right|^2\right]\right) \\
&+ C\mathbb{E}\int_{T'}^T \lambda\bar{\nu}_s ds + C\mathbb{E}\left[\sup_{T'\leq s\leq T}\left|\bar{X}_s^{T',x}\right|^2 \int_{T'}^T \lambda\bar{\nu}_s ds\right] \\
&\leq C(1+\lambda)|T-T'|\left(1 + \mathbb{E}\left[\sup_{T'\leq s\leq T}\left|\bar{X}_s^{T',x}\right|^2\right]\right).
\end{aligned}$$

By (H_X), Doob's inequality and Gronwall's lemma, we also have

$$\mathbb{E}\left[\sup_{T'\leq s\leq T}\left|\bar{X}_s^{T',x}\right|^2\right] \leq C\left(1+x^2\right).$$

leading to

$$\left|g(x) - v_T(T',x)\right| \leq C(1+x^2)^{\frac{1}{2}}|T-T'|^{\frac{1}{2}}.$$

Together with estimate (2.3), we get:

$$\mathbb{E}\left|g(X_{T'}^{u^*}) - v_T(T', X_{T'}^{u^*})\right| \leq C\left(1 + \mathbb{E}\left|X_{T'}^{u^*}\right|^2\right)^{\frac{1}{2}}|T-T'|^{\frac{1}{2}} \leq C\left|T-T'\right|^{\frac{1}{2}},$$

for some constant $C < \infty$ which does not depend either on T or T'. This gives $v_T(0,x) - v_{T'}(0,x) \leq \mathbb{E}\left|g(X_{T'}^{u^*}) - v_T(T', X_{T'}^{u^*})\right| \leq C\left|T-T'\right|^{\frac{1}{2}}$ which concludes the proof. □

Chapter 3

A rate of convergence of the error due to penalization

In this chapter, we provide a convergence rate on the error due to penalization for the value process:

$$\mathcal{E}^p(t,x) := \sup_{0\leq t\leq T} \left| v(t,x) - Y_t^{p,t,x} \right|, \quad \forall 0 \leq t \leq T$$

in which v is defined in (1.2) and $Y^{p,t,x}$ is the value process of solution to (2.12) when $X \equiv (X_s^{t,x})_{t\leq s\leq T}$ is the solution starting at x in t to SDE (2.6). For the sake of simplicity in computations, we deal here with the distance between $Y_0^{p,0,x}$ and $v(0,x)$, that is

$$\mathcal{E}_0^p := \left| v(0,x) - Y_0^{p,0,x} \right|$$

but all the computations hold for any starting time $t \in [0,T]$, so that our estimate for $\mathcal{E}_0^p$ holds for $\mathcal{E}^p$ as well. We obtain a convergence rate of order $(\lambda p)^{\alpha-\frac{1}{2}}, \forall \alpha \in (0, \frac{1}{2})$, see Theorem 3.2.1, in which we recall that p denotes the penalization coefficient and λ, the jump intensity. This result is new and underlines the fact that in practice the jump intensity λ might play a crucial role in the convergence of the approximation by penalization.

3.1 Preliminary results

To compute an estimate for the distance $\mathcal{E}_0^p = \left| v(0,x) - Y_0^{p,0,x} \right|$ as p tends to $+\infty$, our idea is the following: using a representation for the process $(Y_t^{p,0,x})_{0\leq t\leq T}$ following from (2.13), it is possible to give a more tractable bound to $\mathcal{E}_0^p$. By the convenient change of measure presented in Lemma 3.1.1, we will intuitively force the penalized solution to jump as soon as possible after that an optimal impulse happens.

Because of such an approximation of optimal strategy *by the right-hand side*, this choice of measure change might fail to capture the possibility of an optimal impulse date coinciding with maturity T. To deal with this difficulty, we will look at the distance between the penalized solution and the solution to problem (1.2) restricted to strategies with values at finite distance $\eta > 0$ to maturity T. We will conclude by sending η to 0 together with a continuity argument of the value function $v(0,x)$ in its maturity variable, see Proposition 2.3.1.

For any $0 < \eta \leq T$, let us consider problem

$$v_T^\eta(0,x) = \sup_{u=(\tau_k)_{k\geq 1}\in\mathcal{U}_{(0,T-\eta]}} \mathbb{E}\left[g(X_T^u) + \int_0^T f(X_s^u)ds + \sum_{k\geq 1, \tau_k \leq T} \kappa(X^u_{\tau_k^-}) \right] \tag{3.1}$$

which corresponds to initial problem (1.2) at time $t = 0$ restricted to the sub-set of strategies taking values in $(0, T-\eta]$. One can always find an $\eta^{\frac{1}{2}}$-optimal strategy to problem (3.1), namely a strategy

$$u^{\eta*} = (\tau_k^{\eta*})_{k\geq 1} \in \mathcal{U}_{(0,T-\eta]}$$

such that

$$0 \leq v_T^\eta(0,x) - \tilde{v}_T^\eta(0,x) < \eta^{\frac{1}{2}} \tag{3.2}$$

and

$$\tilde{v}_T^\eta(0,x) = \mathbb{E}\left[g(X_T^{u^{\eta*}}) + \int_0^T f(X_s^{u^{\eta*}})ds + \sum_{k\geq 1, \tau_k^{\eta*}\leq T} \kappa(X^{u^{\eta*}}_{\tau_k^{\eta*-}})\right]. \tag{3.3}$$

The following lemma gives a more tractable bound to error $\mathcal{E}_0^p$ involving the penalized solution $Y^{p,0,x}$ modified by a change of measure with respect to the sequence of impulse dates $(\tau_k^{\eta*})_{k\geq 1}$. This measure change affects only the jump part of $Y^{p,0,x}$ and makes it jump as close as possible after that a $\tau_k^{\eta*}$ occurs.

Lemma 3.1.1. Assume (H), (H_1') and (H_2). Then for any $p > 0$,

$$0 \leq \mathcal{E}_0^p \leq v(0,x) - \tilde{Y}_0^p$$

in which

$$\tilde{Y}_0^p := \mathbb{E}\left[g(X_T^p) + \int_0^T f(X_t^p)dt + \sum_{k\geq 1, \tau_k^p\leq T} \kappa(X^p_{\tau_k^{p-}})\right] \tag{3.4}$$

and $(\tau_k^p)_{k\geq 1}$ is the sequence of jump dates of X^p satisfying

$$X_t^p = x + \int_0^t b(X_s^p)ds + \int_0^t \sigma(X_s^p)dW_s + \sum_{k\geq 1, \tau_k^p\leq t} \gamma(X^p_{\tau_k^{p-}}), \quad \forall 0 \leq t \leq T$$

such that

$$\forall s \geq 0, \quad \mathbb{P}\big(\tau_k^p - \tau_k^{\eta*} > s | \sum_{j\geq 1} \mathbb{1}_{\{\tau_j^p \leq \tau_k^{\eta*}\}} = k-1\big) = e^{-\lambda p s}. \tag{3.5}$$

In other words, the increment $\tau_k^p - \tau_k^{\eta*}$, has an exponential distribution[1] with parameter (λp), conditionally to the fact that X^p has jumped *one time less* than $X^{u^{\eta*}}$.

Proof. By monotone convergence of $(Y^{p,0,x})_p$ to $Y^{0,x}$ (see Proposition 2.2.2) and identification $v(0,x) = Y_0^{0,x}$ from Corollary 2.2.1, we have

$$\forall p > 0, \quad \mathcal{E}_0^p = v(0,x) - Y_0^{p,0,x} = Y_0^{0,x} - Y_0^{p,0,x} \geq 0.$$

Let $p > 0$. For any $\mathcal{P}$-measurable essentially bounded positive process ν^p, consider the probability measure $\mathbb{P}^{\nu^p}$ defined on $(\Omega, \mathcal{F}_T)$ by Radon-Nikodym density:

$$\left.\frac{d\mathbb{P}^{\nu^p}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = e^{-\int_0^T(\nu_s^p - 1)\lambda ds} e^{\int_0^T \ln(\nu_s^p)dN_s}$$

[1] Recall that the density function of the exponential law with parameter Λ is $f_{\exp}(x) = \Lambda e^{-\Lambda x}\mathbb{1}_{\{x\geq 0\}}$.

This defines an absolutely continuous probability measure change from $\mathbb{P}$ to $\mathbb{P}^{\nu^p}$. Under $\mathbb{P}^{\nu^p}$, the Brownian motion W remains unchanged whereas N has a (stochastic) intensity $(\lambda\nu_s^p)_{s\geq 0}$. Let us denote by N^p, the doubly stochastic Poisson process (Cox process) with intensity $(\lambda\nu_s^p)_{s\geq 0}$ under $\mathbb{P}$ and by $(\tau_k^p)_{k\geq 1}$ the sequence of its jump dates. Consider then:

$$\forall s \geq 0, \;\; \nu_s^p = \begin{cases} p & \text{if } \sum_{k\geq 1} \mathbb{1}_{\{\tau_k^{\eta^*} < s\}} \neq N_s^p, \\ 0 & \text{else.} \end{cases} \tag{3.6}$$

ν^p is a $\mathcal{P}$-measurable process bounded by p a.s. By definition of the counting process N^p, for any $k \geq 1$, the conditional distribution of $\tau_k^p - \tau_k^{\eta^*}$ given that $N^p_{\tau_k^{\eta^*}} = k-1$ is exponential with a parameter (λp). Hence (3.5).

Recall that X is the solution starting at x in 0 to SDE (2.6). By writing that $(Y^{p,0,x}, Z^{p,0,x}, V^{p,0,x})$ is solution to (2.12) and taking the expectation under $\mathbb{P}^{\nu^p}$, we get

$$\begin{aligned}
Y_0^{p,0,x} &= \mathbb{E}^{\nu^p}\left[g(X_T) + \int_0^T f(X_t)dt + \int_0^T \kappa(X_{t^-})dN_t - \int_0^T Z_t^{p,0,x} dW_t \right. \\
&\quad \left. + p\int_0^T \left(V_t^{p,0,x} + \kappa(X_{t^-})\right)^+ \lambda ds - \int_0^T \left(V_t^{p,0,x} + \kappa(X_{t^-})\right) dN_t \right] \\
&= \mathbb{E}^{\nu^p}\left[g(X_T) + \int_0^T f(X_t)dt + \int_0^T \kappa(X_{t^-})dN_t \right] \\
&\quad + \mathbb{E}\Big[\int_0^T \underbrace{\left\{ p\left(V_t^{p,0,x} + \kappa(X_{t^-}^p)\right)^+ - \nu_t^p \left(V_t^{p,0,x} + \kappa(X_{t^-}^p)\right) \right\}}_{\geq 0} \lambda dt \Big] \\
&\geq \mathbb{E}\left[g(X_T^p) + \int_0^T f(X_t^p)dt + \int_0^T \kappa(X_{t^-}^p) dN_t^p \right] = \tilde{Y}_0^p.
\end{aligned}$$

if X^p denotes the solution to (2.6) driven by N^p and by definition of $\tilde{Y}_0^p$ in (3.4).

□

As a by-product, Lemma 3.1.2 links the penalization error $\mathcal{E}_0^p$ to the distance $\left|\tilde{v}_T^\eta(0,x) - \tilde{Y}_0^p\right|$, whose estimation as p goes to infinity will be performed in the rest of the chapter.

Lemma 3.1.2. Assume (H), (H_1') and (H_2). Then for any $p > 0$ and $0 < \eta \leq T$:

$$\mathcal{E}_0^p \;\leq\; C\eta^{\frac{1}{2}} + \left|\tilde{v}_T^\eta(0,x) - \tilde{Y}_0^p\right|$$

in which $\tilde{v}_T^\eta(0,x)$ is defined in (3.3) and $\tilde{Y}_0^p$ in (3.4), for some constant $C < \infty$ which does not depend either on p or η.

Proof. Let $0 < \eta \leq T$. With Lemma 3.1.1, the penalization error is such that:

$$\begin{aligned}
0 \;\leq\; \mathcal{E}_0^p \;&\leq\; v(0,x) - \tilde{Y}_0^p \\
\leq\; &|v_T(0,x) - v_{T-\eta}(0,x)| + \left|v_{T-\eta}(0,x) - v_T^\eta(0,x)\right| + \left|v_T^\eta(0,x) - \tilde{v}_T^\eta(0,x)\right| + \left|\tilde{v}_T^\eta(0,x) - \tilde{Y}_0^p\right|
\end{aligned}$$

Step 1. The Hölder property of map $\theta \to v_\theta(0,x)$ in Proposition 2.3.1 implies:

$$|v_T(0,x) - v_{T-\eta}(0,x)| \leq C\eta^{\frac{1}{2}}.$$

Besides, by (3.2), the third term is bounded by $\eta^{\frac{1}{2}}$.

Step 2. Let us show that the second term is also of order $\eta^{\frac{1}{2}}$. Recall $u^{\eta*} = (\tau_k^{\eta*})_{k\geq 1} \in \mathcal{U}_{(0,T-\eta]}$ is the $\eta^{\frac{1}{2}}$-optimal strategy of impulse control problem (3.1) giving a value equal to $\tilde{v}_T^\eta(0,x)$ in (3.3). Let $\hat{u} = (\hat{\tau}_k)_{k\geq 1} \in \mathcal{U}_{(0,T-\eta]}$ be the optimal strategy of the impulse control problem whose value function is $v_{T-\eta}(0,x)$. We have

$$\begin{aligned} & v_{T-\eta}(0,x) + \mathbb{E}\left[g(X_T^{\hat{u}}) - g(X_{T-\eta}^{\hat{u}}) + \int_{T-\eta}^T f(X_s^{\hat{u}})ds\right] \\ = \;& \mathbb{E}\left[g(X_T^{\hat{u}}) + \int_0^T f(X_s^{\hat{u}})ds + \sum_{k\geq 1, \hat{\tau}_k \leq T} \kappa(X_{\hat{\tau}_k^-}^{\hat{u}})\right] \;\leq\; v_T^\eta(0,x) \end{aligned} \tag{3.7}$$

by sub-optimality of $\hat{u}$ for v_T^η in (3.1). Conversely:

$$\begin{aligned} & \tilde{v}_T^\eta(0,x) - \mathbb{E}\left[g(X_T^{u^{\eta*}}) - g(X_{T-\eta}^{u^{\eta*}}) + \int_{T-\eta}^T f(X_s^{u^{\eta*}})ds\right] \\ = \;& \mathbb{E}\left[g(X_{T-\eta}^{u^{\eta*}}) + \int_0^{T-\eta} f(X_s^{u^{\eta*}})ds + \sum_{k\geq 1, \tau_k^{\eta*} \leq T-\eta} \kappa(X_{\tau_k^{\eta*-}}^{u^{\eta*}})\right] \;\leq\; v_{T-\eta}(0,x) \end{aligned}$$

by sub-optimality of $u^{\eta*}$ for $v_{T-\eta}$, leading with (3.2) to

$$\begin{aligned} & v_T^\eta(0,x) - v_{T-\eta}(0,x) \;=\; \left[v_T^\eta(0,x) - \tilde{v}_T^\eta(0,x)\right] + \left[\tilde{v}_T^\eta(0,x) - v_{T-\eta}(0,x)\right] \\ \leq \;& \eta^{\frac{1}{2}} + \mathbb{E}\left[g(X_T^{u^{\eta*}}) - g(X_{T-\eta}^{u^{\eta*}}) + \int_{T-\eta}^T f(X_s^{u^{\eta*}})ds\right]. \end{aligned} \tag{3.8}$$

Inequalities (3.7) and (3.8) imply

$$\begin{aligned} & \left|v_T^\eta(0,x) - v_{T-\eta}(0,x)\right| \\ \leq \;& \eta^{\frac{1}{2}} + \max_{u\in\{u^{\eta*},\hat{u}\}} \left|\mathbb{E}\left[g(X_T^u) - g(X_{T-\eta}^u) + \int_{T-\eta}^T f(X_s^u)ds\right]\right| \\ \leq \;& \eta^{\frac{1}{2}} + \max_{u\in\{u^{\eta*},\hat{u}\}} \mathbb{E}\left|g(X_T^u) - g(X_{T-\eta}^u) + \int_{T-\eta}^T f(X_s^u)ds\right| \\ \leq \;& \eta^{\frac{1}{2}} + C \max_{u\in\{u^{\eta*},\hat{u}\}} \left\{\mathbb{E}\left|X_T^u - X_{T-\eta}^u\right| + \mathbb{E}\left[\int_{T-\eta}^T (1+|X_s^u|)\,ds\right]\right\} \\ \leq \;& \eta^{\frac{1}{2}} + C \max_{u\in\{u^{\eta*},\hat{u}\}} \left\{\left(\mathbb{E}\left|X_T^u - X_{T-\eta}^u\right|^2\right)^{\frac{1}{2}} + \eta\left[1 + \left(\mathbb{E}\left[\sup_{T-\eta\leq s\leq T}|X_s^u|^2\right]\right)^{\frac{1}{2}}\right]\right\} \end{aligned}$$

by Lipschitz property of g, linear growth of f and Jensen's inequality. Let $u \in \{u^{\eta*}, \hat{u}\}$. $(\tilde{X}_t)_{t\geq T-\eta} \equiv (X_t^u)_{t\geq T-\eta}$ is an uncontrolled process with dynamics

$$\tilde{X}_t = X_{T-\eta}^u + \int_{T-\eta}^t b(\tilde{X}_s)ds + \int_{T-\eta}^t \sigma(\tilde{X}_s)dW_s, \quad \forall t \geq T-\eta.$$

A straightforward computation using (H_X), Doob's inequality, and Gronwall's lemma shows that

$$\begin{aligned} \mathbb{E}\left[\sup_{T-\eta\leq s\leq T}|\tilde{X}_s|^2\right] &\leq C\left(1 + \mathbb{E}|X_{T-\eta}^u|^2\right) \\ \mathbb{E}\left|\tilde{X}_T - \tilde{X}_{T-\eta}\right|^2 &\leq C\left(1 + \mathbb{E}|X_{T-\eta}^u|^2\right)\eta \end{aligned}$$

As $u \in \mathcal{U}_{(0,T-\eta]}$, estimate (2.3) gives finally

$$\left|v_T^\eta(0,x) - v_{T-\eta}(0,x)\right| \leq C\eta^{\frac{1}{2}}$$

for some constant $C < \infty$ which does not depend either on p or η. □

3.2 Convergence rate of the approximation by penalization

We are reduced to estimate the distance $\left|\tilde{v}_T^\eta(0,x) - \tilde{Y}_0^p\right|$ such that

$$\begin{aligned}
&\tilde{v}_T^\eta(0,x) - \tilde{Y}_0^p \\
&= \mathbb{E}\left[g(X_T^{u^*}) + \int_0^T f(X_s^{u^*})ds + \sum_{\tau_k^{\eta^*} \leq T-\eta} \kappa(X_{\tau_k^{*-}}^{u^*})\right] - \mathbb{E}\left[g(X_T^p) + \int_0^T f(X_s^p)ds + \sum_{\tau_k^p \leq T} \kappa(X_{\tau_k^{p-}}^p)\right] \\
&= \mathbb{E}\left[g(X_T^{u^{\eta^*}}) - g(X_T^p) + \int_0^T \Big(f(X_s^{u^{\eta^*}}) - f(X_s^p)\Big)ds + \sum_{\tau_k^{\eta^*} \leq T-\eta} \kappa(X_{\tau_k^{*-}}^{u^{\eta^*}}) - \sum_{\tau_k^p \leq T} \kappa(X_{\tau_k^{p-}}^p)\right]
\end{aligned} \tag{3.9}$$

in which

$$\forall t \geq 0, \quad X_t^{u^{\eta^*}} = x + \int_0^t b(X_s^{u^{\eta^*}})ds + \int_0^t \sigma(X_s^{u^{\eta^*}})dW_s + \sum_{\tau_k^{\eta^*} \leq t} \gamma(X_{\tau_k^{\eta^{*-}}}^{u^{\eta^*}}) \tag{3.10}$$

$$X_t^p = x + \int_0^t b(X_s^p)ds + \int_0^t \sigma(X_s^p)dW_s + \sum_{\tau_k^p \leq t} \gamma(X_{\tau_k^{p-}}^p) \tag{3.11}$$

and the sequence $(\tau_k^p)_{k\geq 1}$ is defined in Lemma 3.1.1 with respect to $u^{\eta^*} = (\tau_k^{\eta^*})_{k\geq 1}$. Let n^{η^*} be the number of impulses in strategy u^{η^*}

$$n^{\eta^*} := \sharp\left\{k \geq 1 : \tau_k^{\eta^*} \leq T - \eta\right\}.$$

We introduce the following assumptions on impulse strategy u^{η^*}:

(H^n) There exists some $\bar{n} \in \mathbb{N}^*$ such that

$$\forall j \geq \bar{n}, \quad \mathbb{P}\left(n^{\eta^*} \geq j\right) \leq l(j)$$

for some map l such that

$$l(j) \leq e^{-Cj} \text{ for some some constant } C > 0.$$

(H*) There exists a map h such that $h(\epsilon) = \mathcal{O}_{\epsilon\to 0}(\epsilon^{\frac{1}{2}})$ and

$$\forall \epsilon > 0, \quad \mathbb{P}\left\{H^{*,\epsilon}\right\} := \mathbb{P}\left\{\min_{k\geq 1}\left|\tau_{k+1}^{\eta^*} - \tau_k^{\eta^*}\right| \leq \epsilon\right\} \leq h(\epsilon).$$

Assumption (H^n) means that the right tail of the distribution of $n^{\eta*}$ decreases faster than exponentially. Intuitively, (H^n) might be verified as soon as the controlled state variable is constrained almost surely and admits jumps of constant sign, see for example in Ly Vath et al. [85] for this kind of assumptions in the framework of optimal portfolio selection under transaction costs. We provide below an example for which (H^n) is satisfied, see Example 3.2.1.

Assumption (H^*) should be understood as follows: two consecutive impulse dates in strategy $u^{\eta*}$ are close with a small probability. If there is some time delay between two consecutive interventions, this assumption is automatically satisfied.

Besides, in practice, we solve the discrete-time counterpart to impulse control problem (1.2), meaning that we restrict ourselves to the sub-set of admissible strategies taking values in some time grid. If this time grid is regular, let δ be the discrete-time step and $\mathcal{U}^{\delta}_{(t,T]}$ the sub-set of strategies such that the distance between two consecutive impulse dates is bigger that δ. The same kind of control on the penalization error as in Theorem 3.2.1 might be proven if we could show that the distance between the solution to (1.2) and the solution to (1.2) restricted to $\mathcal{U}^{\delta}_{(t,T]}$ is of order $\delta^{\frac{1}{2}}$. Such a result might be easily shown in the framework of multiple optimal stopping by adaptating the same arguments as Bally and Pagès [5]. However, in our framework of impulse control (with a *controlled* state variable), this seems to be a difficult problem.

Remark 3.2.1. Both assumptions (H^n) and (H^*) are satisfied in practical examples studied in Chapter 5.

Example 3.2.1. Let us consider impulse control problem (1.2) with a state variable defined in (1.1) such that

- b is uniformly bounded and $\sigma > 0$ constant
- For some constant $c > 0$,
$$\sup_{x\in\mathbb{R}^d} \gamma(x) \leq -c.$$

Assume that the optimal strategy u^* to (1.2) implies $X_T^{u^*} \geq 0$ a.s. Then the number of optimal impulses $n^*_{(0,T]}$ satisfies, for any $a > 0$,

$$\mathbb{P}\left(n^*_{(0,T]} > n\right) = \mathcal{O}\left(e^{-an}\right) \quad \text{as } n \to +\infty.$$

Proof. We have

$$\begin{aligned} & X_T^{u^*} = x + \int_0^T b(X_s^{u^*})ds + \sigma W_T + \sum_{\tau_k^* \leq T} \gamma(X^{u^*}_{\tau_k^{*-}}) \geq 0 \\ \iff \quad & n^*_{(0,T]} \leq \frac{1}{c}\left(x + \int_0^T b(X_s^{u^*})ds + \sigma W_T\right) \\ \implies \quad & n^*_{(0,T]} \leq C\left(x + T + W_T\right) \end{aligned}$$

for some constant $C < \infty$. In consequence, for any $n \geq 1$,

$$\begin{aligned}\mathbb{P}\left(n^*_{(0,T]} > n\right) &\leq \mathbb{P}\left[C\left(x + T + W_T\right) > n\right] \\ &\leq e^{-an}\mathbb{E}\left[e^{aC(x+T+W_T)}\right] \\ &= \exp\left\{aC(x+T) + \frac{T}{2}(aC)^2\right\} e^{-an}\end{aligned}$$

and this holds for any $a > 0$. □

Our computation of $\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right|$ is based on an iteration on the indexes of jump dates $\tau_k^{\eta*}$. Indeed, as p goes to infinity, the jump diffusion X^p in (3.11) tends to mimic the dynamics of controlled state variable $X^{u^{\eta*}}$ in (3.10), since each τ_k^p becomes closer to its corresponding $\tau_k^{\eta*}$, recall (3.5). By recursion, we shall thus handle with quantities involving tractable increments as $|\tau_k^p - \tau_k^{\eta*}|$, see Figure 3.1.

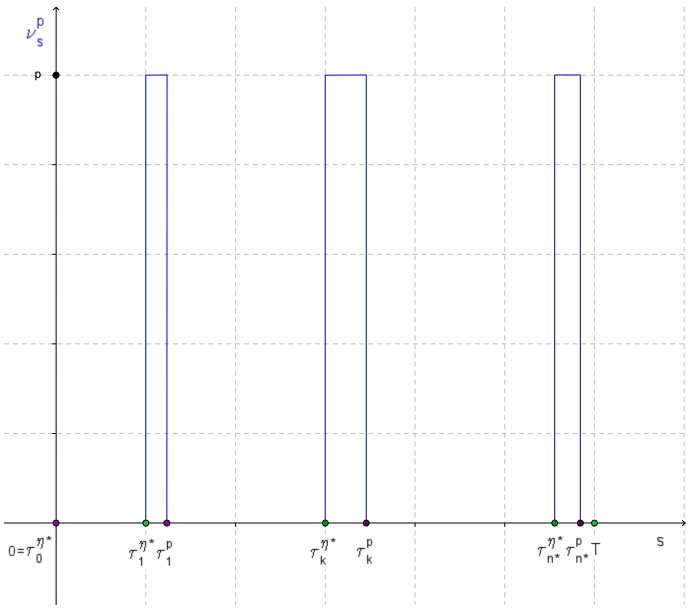

Figure 3.1: Change of measure introduced to perform the estimation of error $\mathcal{E}_0^p$ on event A^{*p}.

Such an inductive reasoning is conceivable if the jump dates $(\tau_k^p)_{k\geq 1}$ of jump diffusion X^p are *well-drawn with respect to the* $(\tau_k^{\eta*})_{k\geq 1}$, namely that they insert in between the sequence of impulse dates $(\tau_k^{\eta*})_{k\geq 1}$. Following this remark, introduce the event

$$A^{*p} \;=\; \left\{(\tau_k^p)_{k\geq 1} : 0 < \tau_1^{\eta*} < \tau_1^p \leq \ldots \leq \tau_k^{\eta*} < \tau_k^p \leq \ldots \leq \tau_{n^{\eta*}}^{\eta*} < \tau_{n^{\eta*}}^p \leq T\right\}, \tag{3.12}$$

whose probability is intuitively close to 1 for a big enough penalization coefficient p. The following lemma gives more rigorously a decomposition of the distance $\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right|$ with respect to A^{*p}.

Lemma 3.2.1. Let assumption (H*) be satisfied and assume that $n^{\eta*} \in \mathrm{L}^\infty$ meaning that for some $n \in \mathbb{N}^*$, $n^{\eta*} \leq n$ almost surely. Then, for any $p > 0$ and $0 < \eta \leq T$,

$$\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right| = \mathcal{E}_0^{p,\eta,n}\ \mathbb{P}\left\{A^{*p}\right\} + \hat{\mathcal{E}}_0^{p,\eta,n}\ \mathbb{P}\left\{\complement A^{*p}\right\},$$

where $\mathcal{E}_0^{p,\eta,n} = \mathbb{E}\left[|\tilde{v}^\eta(0,x) - Y_0^p|\,|A^{*p}\right]$ and $\hat{\mathcal{E}}_0^{p,\eta,n} = \mathbb{E}\left[|\tilde{v}^\eta(0,x) - Y_0^p|\,|\complement A^{*p}\right]$ and

$$\begin{aligned}\forall \epsilon > 0, \quad \mathbb{P}\left\{A^{*p}\right\} &\geq h_{\lambda,n}(\epsilon,\eta)\\ \mathbb{P}\left\{\complement A^{*p}\right\} &\leq 1 - h_{\lambda,n}(\epsilon,\eta),\end{aligned}$$

where

$$h_{\lambda,n}(\epsilon,\eta) := \left(1-(n-1)e^{-\lambda p\epsilon}\right)\left(1-e^{-\lambda p\eta}\right)(1-h(\epsilon)) \longrightarrow 1$$

as soon as $\epsilon \to 0, \lambda p\epsilon \to +\infty, \lambda p\eta \to +\infty$.

Proof. We have:

$$\begin{aligned}\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right| &= \mathbb{E}\left[\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right| \mathbb{1}_{\{A^{*p}\}} + \left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right| \mathbb{1}_{\left\{\complement(A^{*p})\right\}}\right]\\ &= \mathcal{E}_0^{p,\eta,n}\ \mathbb{P}\left\{A^{*p}\right\} + \hat{\mathcal{E}}_0^{p,\eta,n}\ \mathbb{P}\left\{\complement(A^{*p})\right\}.\end{aligned}$$

Let $\epsilon > 0$. We have

$$\begin{aligned}\mathbb{P}\left\{A^{*p}\right\} &\geq \mathbb{P}\left\{A^{*p} \cap \complement H^{*,\epsilon} \cap (n^{\eta*} \leq n)\right\}\\ &= \mathbb{P}\left\{\left(\forall 1 \leq k \leq n, \tau_k^p \in (\tau_k^{\eta*}, \tau_{k+1}^{\eta*}]\right) \cap \left(\forall k \geq 1, \tau_{k+1}^{\eta*} - \tau_k^{\eta*} > \epsilon\right)\right\}\\ &= \prod_{k=1}^{n-1} \mathbb{P}\left\{\left(\tau_k^p \in (\tau_k^{\eta*}, \tau_{k+1}^{\eta*}]\right) \cap \left(\tau_{k+1}^{\eta*} - \tau_k^{\eta*} > \epsilon\right)\right\} \mathbb{P}\left\{(\tau_n^p \in (\tau_n^{\eta*}, T]) \cap (\tau_n^{\eta*} \leq T - \eta)\right\}.\end{aligned}$$

But:

$$\mathbb{P}\left\{\left(\tau_1^p \in (\tau_1^{\eta*}, \tau_2^{\eta*}]\right) \cap \left(\tau_2^{\eta*} - \tau_1^{\eta*} > \epsilon\right)\right\} \geq \mathbb{P}\left(\tau_1^p - \tau_1^{\eta*} \leq \epsilon\right) = 1 - e^{-\lambda p\epsilon}.$$

Besides, by Lemma 3.1.1), we get, $\forall 2 \leq k \leq n-1$:

$$\mathbb{P}\left\{\left(\tau_k^p \in (\tau_k^{\eta*}, \tau_{k+1}^{\eta*}]\right) \cap \left(\tau_{k+1}^{\eta*} - \tau_k^{\eta*} > \epsilon\right)\right\} \geq \mathbb{P}\big(\tau_k^p - \tau_k^{\eta*} \leq \epsilon | \sum_{j\geq 1} \mathbb{1}_{\left\{\tau_j^p \leq \tau_k^{\eta*}\right\}} = k-1\big) = 1 - e^{-\lambda p\epsilon},$$

and as strategy $u^{\eta*} = (\tau_k^{\eta*})_{k\geq 1}$ belongs to $\mathcal{U}_{(0,T-\eta]}$,

$$\mathbb{P}\left\{(\tau_n^p \in (\tau_n^{\eta*}, T]) \cap (\tau_n^{\eta*} \leq T - \eta)\right\} \geq \mathbb{P}\big(\tau_n^p - \tau_n^{\eta*} \leq \eta | \sum_{j\geq 1} \mathbb{1}_{\left\{\tau_j^p \leq \tau_n^{\eta*}\right\}} = n-1\big) = 1 - e^{-\lambda p\eta}.$$

This leads to:

$$\mathbb{P}\left\{A^{*p}\right\} \geq \left(1-e^{-\lambda p\epsilon}\right)^{n-1}\left(1-e^{-\lambda p\eta}\right)(1-h(\epsilon)) \geq h_{\lambda,n}(\epsilon,\eta).$$

This lower bound tends to 1 when ϵ goes to 0 as soon as $\lambda p\epsilon \to +\infty$ and $\lambda p\eta \to +\infty$. As a by-product, we also get the expected bound for $\mathbb{P}\left\{\complement A^{*p}\right\}$. □

The expression for $\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right|$ given in previous Lemma 3.2.1 is made up of two parts. Sending ϵ to 0, the second term is negligible so that it is sufficient to give a bound to $\hat{\mathcal{E}}_0^{p,\eta,n}$ with order 0 as $p \to +\infty$. On the contrary, the first term is predominant and of the same order as $\mathcal{E}_0^{p,\eta,n}$. Its convergence rate to 0 as $p \to +\infty$ determines the convergence rate of $\left|\tilde{v}^\eta(0,x) - \tilde{Y}_0^p\right|$. Proposition 3.2.1 provide controls on both quantities.

Proposition 3.2.1. *Let assumption* (H) *be satisfied and assume as in Lemma 3.2.1 that* $n^{\eta *} \in \mathrm{L}^\infty$ *with* $n^{\eta *} \leq n$ *almost surely for some* $n \in \mathbb{N}^*$. *Then, for any* $p > 0$ *and* $0 < \eta \leq T$, *errors* $\mathcal{E}_0^{p,\eta,n}$ *and* $\hat{\mathcal{E}}_0^{p,\eta,n}$ *defined in Lemma 3.2.1 are such that*

$$\begin{aligned} \mathcal{E}_0^{p,\eta,n} &\leq B_n^1(\lambda,p)\frac{1}{(\lambda p)^{\frac{1}{2}}} \\ \hat{\mathcal{E}}_0^{p,\eta,n} &\leq B_n^2(\lambda,p) \end{aligned}$$

where:

$$\forall i \in \{1,2\}, \quad B_n^i(\lambda,p) = \mathcal{O}_{p\to+\infty}\left(n\bar{C}^n\right)$$

for some constant $\bar{C} > 1$ *which do not depend either on* λ, p, η *or* n.

Proof. The proof is deferred to Section 3.3. □

Remark 3.2.2. It is clear from the computations that are performed in all the proofs that all the results of this chapter hold if we let the maps b, σ, γ, f, κ and g depend on t, as soon as these functions are $\frac{1}{2}$-Hölder in t and the assumption (H) is satisfied uniformly in t.

The results presented in current and previous sections finally lead to the following estimate of the penalization error $\mathcal{E}_0^p$.

Theorem 3.2.1. *Let assumptions* (H), (H^n), (H^*), (H_1') *and* (H_2) *be satisfied. Then the penalization error admits the following bound:*

$$\mathcal{E}_0^p \leq \bar{n}\bar{C}^{\bar{n}}\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}}, \quad \forall \alpha \in \left(0,\frac{1}{2}\right),$$

for some constant $\bar{C} > 1$, *which do not depend either on* λ, p, η *or* $\bar{n}$.

Proof. Set here for the sake of simplicity in notations

$$\mathcal{E}_0^p := \mathbb{E}\left[\Phi\left(n^{\eta *}, X^p, X^{u^{\eta *}}\right)\right]$$

in which

$$\begin{aligned} \Phi\left(n^{\eta *}, X^p, X^{u^{\eta *}}\right) = g(X_T^{u^{\eta *}}) - g(X_T^p) + \int_0^T \left(f(X_s^{u^{\eta *}}) - f(X_s^p)\right) ds \\ + \sum_{k\geq 1, \tau_k^{\eta *} \leq T-\eta} \kappa(X^{u^{\eta *}}_{\tau_k^{*-}}) - \sum_{k\geq 1, \tau_k^p \leq T} \kappa(X^p_{\tau_k^{p-}}). \end{aligned}$$

Let us first deal with the case when $n^{\eta *} \in \mathrm{L}^\infty$. In a second step, the same result will be shown to hold under weaker condition (H^n) as well, by a simple conditioning argument.
Step 1. If $n^{\eta *} \in \mathrm{L}^\infty$, there exists some $n \in \mathbb{N}^*$ such that $n^{\eta *} \leq n$ a.s. and

$$\mathcal{E}_0^p = \mathcal{E}_0^p(n) := \mathbb{E}\left[\Phi\left(n^{\eta *}, X^p, X^{u^{\eta *}}\right) | n^{\eta *} \leq n\right].$$

Lemma 3.1.2, Lemma 3.2.1 and Proposition 3.2.1 imply, for any $p > 0$, $0 < \eta \leq T$ and $\epsilon > 0$:

$$\mathcal{E}_0^p(n) \leq C\eta^{\frac{1}{2}} + B_n^1(\lambda,p)\frac{1}{(\lambda p)^{\frac{1}{2}}} + B_n^2(\lambda,p)\left(1 - h_{\lambda,n}(\epsilon,\eta)\right).$$

The right hand side of previous inequality is equivalent as $p \to +\infty$, $\epsilon \to 0$, $\lambda p\epsilon \to +\infty$ and $\lambda p\eta \to +\infty$ to

$$C\eta^{\frac{1}{2}} + Cn\bar{C}^n \frac{1}{(\lambda p)^{\frac{1}{2}}} + n\bar{C}^n h(\epsilon).$$

Choosing η, ϵ with respect to (λ, p) as

$$\eta = \epsilon = \frac{1}{(\lambda p)^{1-2\alpha}}, \quad \forall \alpha \in \left(0, \frac{1}{2}\right)$$

leads to

$$\mathcal{E}_0^p(n) = \mathcal{O}_{p\to+\infty}\left(n\bar{C}^n \frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}}\right)$$

for a different constant $\bar{C} > 1$.

Step 2. Let assumption (H^n) be satisfied. Conditioning the error $\mathcal{E}_0^p$ in relation to (H^n), we get

$$\mathcal{E}_0^p \leq \underbrace{\mathbb{E}\left[\Phi\left(n^{\eta*}, X^p, X^{u^{\eta*}}\right) \mid n^{\eta*} \leq \bar{n}\right]}_{:= \mathcal{E}_0^p(\bar{n})} + \underbrace{\mathbb{E}\left[\Phi\left(n^{\eta*}, X^p, X^{u^{\eta*}}\right) \mathbb{1}_{\{n^{\eta*} > \bar{n}\}}\right]}_{:= \bar{\mathcal{E}}_0^p}.$$

Step 1 allows to control $\mathcal{E}_0^p(\bar{n})$ which corresponds to the penalization error when the number of impulses of strategy $u^{\eta*}$ is almost surely bounded by $\bar{n}$. This gives

$$\mathcal{E}_0^p(\bar{n}) = \mathcal{O}_{p\to+\infty}\left(\bar{n}\bar{C}^{\bar{n}} \frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}}\right), \quad \forall \alpha \in \left(0, \frac{1}{2}\right).$$

Let us now deal with error $\bar{\mathcal{E}}_0^p$. The family of integer intervals $\left([2^n, 2^{n+1})\right)_{n\in\mathbb{N}}$ forms a partition of $\mathbb{N}^*$. Set $\bar{n}_0 := \max\{n \in \mathbb{N}, 2^n \leq \bar{n} + 1\} = \lfloor \ln(\bar{n}+1)/\ln 2 \rfloor$. Then

$$\begin{aligned}
\bar{\mathcal{E}}_0^p &= \mathbb{E}\left[\Phi\left(n^{\eta*}, X^p, X^{u^{\eta*}}\right) \sum_{j=\bar{n}+1}^{+\infty} \mathbb{1}_{\{n^{\eta*}=j\}}\right] \\
&\leq \mathbb{E}\left[\Phi\left(n^{\eta*}, X^p, X^{u^{\eta*}}\right) \sum_{n=\bar{n}_0}^{+\infty} \mathbb{1}_{\{2^n \leq n^{\eta*} < 2^{n+1}\}}\right] \\
&= \sum_{n=\bar{n}_0}^{+\infty} \mathbb{E}\left[\Phi\left(X^p, X^{u^{\eta*}}, n^{\eta*}\right) \mid 2^n \leq n^{\eta*} < 2^{n+1}\right] \mathbb{P}\left(2^n \leq n^{\eta*} < 2^{n+1}\right) \\
&\leq \sum_{n=\bar{n}_0}^{+\infty} \mathbb{E}\left[\Phi\left(X^p, X^{u^{\eta*}}, n^{\eta*}\right) \mid n^{\eta*} \leq 2^{n+1}\right] \mathbb{P}\left(n^{\eta*} \geq 2^n\right) \\
&= \sum_{n=\bar{n}_0}^{+\infty} \mathcal{E}_0^p(2^{n+1})\, l(2^n)
\end{aligned}$$

where $\mathcal{E}_0^p(2^{n+1})$ denotes the penalization error when the number of impulses $n^{\eta*}$ is almost surely bounded by 2^{n+1}. Again, Step 1 implies

$$\bar{\mathcal{E}}_0^p \leq \frac{C}{(\lambda p)^{\frac{1}{2}-\alpha}} \sum_{n=\bar{n}_0}^{+\infty} 2^n \left(\bar{C}^2\right)^{2^n} l(2^n), \quad \forall \alpha \in \left(0, \frac{1}{2}\right)$$

in which $C > 0$ is a constant independent of λ, p, η and $\bar{n}$. By definition of l in (H^n), this leads to

$$\bar{\mathcal{E}}_0^p \leq \frac{C}{(\lambda p)^{\frac{1}{2}-\alpha}}$$

for a different constant C, which concludes the proof. □

3.3 Proof of Proposition 3.2.1

Throughout this section, we fix $0 < \eta \leq T$, suppose that assumption (H) is satisfied and that $n^{\eta *} \in \mathrm{L}^\infty$ meaning that for some $n \in \mathbb{N}^*$,

$$n^{\eta *} \leq n \quad \text{a.s.}$$

For the sake of simplicity in notations in this paragraph, the strategy $u^{\eta *} = (\tau_k^{\eta *})_{k\geq 1}$ related to (3.1) is renamed in $u^* = (\tau_k^*)_{k\geq 1}$ and its number of impulses $n^{\eta *}$ in n^*.

3.3.1 A priori estimates

Introduce first some useful a priori estimates for jump diffusion X^p defined in (3.11).

Lemma 3.3.1. Let $p > 0$. Assume that $(\tau_k^p)_{k\geq 1} \in A^{*p}$, where A^{*p} is defined in (3.12). Then, for any $k \geq 1$:

(i) $\mathbb{E}\left[|X^p_{\tau_k^{p-}}|^4 \mathbb{1}_{\{k\leq n^*\}}\right] \leq A_{x,k}(\lambda, p)$,

(ii) $\mathbb{E}\left[|X^p_{t\wedge\tau_k^{*-}} - X^p_{t\wedge\tau_k^{p-}}|^2 \mathbb{1}_{\{k\leq n^*\}}\right] \leq C A_{x,k}(\lambda, p)\left(\frac{1}{\lambda p} + \frac{1}{(\lambda p)^2}\right), \ \forall t \in [0, T]$,

where

$$A_{x,k}(\lambda, p) = \mathcal{O}_{p\to+\infty}\left(C^k e^{C^k}\left(1 + x^4\right)\right),$$

for some constant $C > 0$ which does not depend either on λ, p, η or x.

Proof. The computation of these bounds are principally based on iterated estimations on the time intervals $[\tau_{k-1}^p, \tau_k^p)$, $\forall k \geq 1$, Lipschitz property of b, σ and γ in (H_X), the use of Gronwall's lemma and the conditional law of $\tau_k^p - \tau_k^*$ (see Lemma 3.1.1) which implies in particular that

$$\begin{aligned}\forall m \in \mathbb{N}, \quad \mathbb{E}\left[\left|\tau_k^p - \tau_k^*\right|^m\right] &= \mathbb{E}\Big[\left|\tau_k^p - \tau_k^*\right|^m \Big| \sum_{j\geq 1} \mathbb{1}_{\{\tau_j^p \leq \tau_k^{\eta *}\}} = k-1\Big] \\ &= \int_0^\infty x^m \lambda p e^{-\lambda p x} dx = \frac{m!}{(\lambda p)^m},\end{aligned}$$

recalling that $(\tau_k^p)_{k\geq 1} \in A^{*p}$.
Step 1. Let $k \geq 2$ and $s \in [\tau_{k-1}^p, \tau_k^p)$ we have

$$X_s^p = X^p_{\tau_{k-1}^{p-}} + \gamma(X^p_{\tau_{k-1}^{p-}}) + \int_{\tau_{k-1}^p}^s \left[b(X_r^p)dr + \sigma(X_r^p)dW_r\right].$$

Let $t \in [0,T]$. Multiplying previous expression by $\mathbb{1}_{\{k\leq n^*\}}$, raising to power 4, taking the supremum on $[t\wedge\tau_{k-1}^p, t\wedge\tau_k^p]$ and the expectation, we get

$$\begin{aligned}
&\mathbb{E}\left[|X^p_{t\wedge\tau_k^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right] \leq \mathbb{E}\left[\Big(\sup_{t\wedge\tau_{k-1}^p\leq s\leq t\wedge\tau_k^p}|X_s^p|^4\Big)\mathbb{1}_{\{k\leq n^*\}}\right]\\
&\leq C\mathbb{E}|X^p_{t\wedge\tau_{k-1}^{p-}}\mathbb{1}_{\{k\leq n^*\}}|^4 + C\mathbb{E}|\gamma(X^p_{t\wedge\tau_{k-1}^{p-}})\mathbb{1}_{\{k\leq n^*\}}|^4\\
&+ C\mathbb{E}\left[\sup_{t\wedge\tau_{k-1}^p\leq s\leq t\wedge\tau_k^p}|\int_{t\wedge\tau_{k-1}^p}^s b(X_r^p)\mathbb{1}_{\{k\leq n^*\}}dr|^4\right] + C\mathbb{E}\left[\sup_{t\wedge\tau_{k-1}^p\leq s\leq t\wedge\tau_k^p}|\int_{t\wedge\tau_{k-1}^p}^s \sigma(X_r^p)\mathbb{1}_{\{k\leq n^*\}}dW_r|^4\right].
\end{aligned}$$

The decomposition $\tau_k^p - \tau_{k-1}^p = (\tau_k^p - \tau_k^*) - (\tau_{k-1}^p - \tau_{k-1}^*) + (\tau_k^* - \tau_{k-1}^*)$ implies

$$\forall m\in\mathbb{N},\quad \mathbb{E}\left|\tau_k^p - \tau_{k-1}^p\right|^m \leq 2C\frac{1}{(\lambda p)^m} + C\mathbb{E}\left|\tau_k^* - \tau_{k-1}^*\right|^m.$$

By assumption (H_X), Burkholder-Davis-Gundy's and Cauchy-Schwartz's inequalities, we get

$$\begin{aligned}
&\mathbb{E}\left[|X^p_{t\wedge\tau_k^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right] \leq \mathbb{E}\left[\Big(\sup_{t\wedge\tau_{k-1}^p\leq s\leq t\wedge\tau_k^p}|X_s^p|^4\Big)\mathbb{1}_{\{k\leq n^*\}}\right]\\
\leq\ & C\mathbb{E}\left[|X^p_{t\wedge\tau_{k-1}^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right] + C\mathbb{E}[\left|\gamma(X^p_{t\wedge\tau_{k-1}^{p-}})\right|^4\mathbb{1}_{\{k\leq n^*\}}]\\
&+ C\mathbb{E}\left[\left|\tau_k^p - \tau_{k-1}^p\right|^3\mathbb{1}_{\{k\leq n^*\}}\right]\mathbb{E}\left[\int_{t\wedge\tau_{k-1}^p}^{t\wedge\tau_k^p}|b(X_r^p)|^4\mathbb{1}_{\{k\leq n^*\}}dr\right] + C\mathbb{E}\left|\int_{t\wedge\tau_{k-1}^p}^{t\wedge\tau_k^p}|\sigma(X_r^p)|^2\mathbb{1}_{\{k\leq n^*\}}dr\right|^2\\
\leq\ & C + C\mathbb{E}\left[|X^p_{t\wedge\tau_{k-1}^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right] + C\left(\frac{1}{(\lambda p)^3}+1\right)\mathbb{E}\left[\int_{t\wedge\tau_{k-1}^p}^{t\wedge\tau_k^p}|b(X_r^p)|^4\mathbb{1}_{\{k\leq n^*\}}dr\right]\\
&+C\mathbb{E}\left[\left|\tau_k^p - \tau_{k-1}^p\right|\mathbb{1}_{\{k\leq n^*\}}\right]\mathbb{E}\left[\int_{t\wedge\tau_{k-1}^p}^{t\wedge\tau_k^p}|\sigma(X_r^p)|^4\mathbb{1}_{\{k\leq n^*\}}dr\right]\\
\leq\ & C + C\mathbb{E}\left[|X^p_{t\wedge\tau_{k-1}^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right] + C\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\mathbb{E}\left[\int_{t\wedge\tau_{k-1}^p}^{t\wedge\tau_k^p}\left(1+|X_r^p|^4\right)\mathbb{1}_{\{k\leq n^*\}}dr\right]\\
\leq\ & C + C\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right) + C\mathbb{E}\left[|X^p_{t\wedge\tau_{k-1}^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right]\\
&+C\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\mathbb{E}\left[\int_{t\wedge\tau_{k-1}^p}^{t\wedge\tau_k^p}|X_r^p|^4\mathbb{1}_{\{k\leq n^*\}}dr\right].
\end{aligned}$$

Actually, the same estimation holds for any $i = 2,\dots,k$, that is

$$\begin{aligned}
\mathbb{E}\left[|X^p_{t\wedge\tau_i^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right] &\leq \mathbb{E}\left[\Big(\sup_{t\wedge\tau_{i-1}^p\leq s\leq t\wedge\tau_i^p}|X_s^p|^4\Big)\mathbb{1}_{\{k\leq n^*\}}\right]\\
&\leq C + C\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right) + C\mathbb{E}\left[|X^p_{t\wedge\tau_{i-1}^{p-}}|^4\mathbb{1}_{\{k\leq n^*\}}\right]\\
&+C\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\mathbb{E}\left[\int_{t\wedge\tau_{i-1}^p}^{t\wedge\tau_i^p}|X_r^p|^4\mathbb{1}_{\{k\leq n^*\}}dr\right]
\end{aligned}$$

and for $i = 1$

$$\mathbb{E}\left[|X^p_{t\wedge\tau_1^{p-}}|^4 \mathbb{1}_{\{k\le n^*\}}\right] \le \mathbb{E}[\big(\sup_{0\le s\le t\wedge\tau_1^p}|X^p_s|^4\big)\mathbb{1}_{\{k\le n^*\}}]$$
$$\le Cx^4 + C\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right) + C\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\mathbb{E}\left[\int_0^{t\wedge\tau_1^p}|X^p_r|^4\mathbb{1}_{\{k\le n^*\}}dr\right].$$

By recursion from $k-1$ to 1, it follows

$$\begin{aligned}\mathbb{E}\left[|X^p_{t\wedge\tau_k^{p-}}|^4\mathbb{1}_{\{k\le n^*\}}\right] &\le \mathbb{E}\left[\big(\sup_{t\wedge\tau^p_{k-1}\le s\le t\wedge\tau^p_k}|X^p_s|^4\big)\mathbb{1}_{\{k\le n^*\}}\right]\\ &\le C^{k-1} + C^{k-1}\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right) + C^{k-1}\mathbb{E}\left[|X^p_{t\wedge\tau_1^{p-}}|^4\mathbb{1}_{\{k\le n^*\}}\right]\\ &+ C^{k-1}\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\sum_{i=1}^{k-1}\mathbb{E}\left[\int_{t\wedge\tau_i^p}^{t\wedge\tau^p_{i+1}}|X^p_r|^4\mathbb{1}_{\{k\le n^*\}}dr\right]\\ &= C^k x^4 + C^{k-1} + C^k\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\\ &+ C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\int_0^t\mathbb{E}\left[|X^p_{r\wedge\tau_k^p}|^4\mathbb{1}_{\{k\le n^*\}}\right]dr. \qquad (3.13)\end{aligned}$$

Applying Gronwall's lemma to $t\to\mathbb{E}\left[|X^p_{t\wedge\tau_k^{p-}}|^4\mathbb{1}_{\{k\le n^*\}}\right]$, we get $\forall t\in[0,T]$

$$\mathbb{E}\left[|X^p_{t\wedge\tau_k^{p-}}|^4\mathbb{1}_{\{k\le n^*\}}\right] \le C^k\big(x^4+1+\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\big)e^{C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)t} \qquad (3.14)$$

which gives (i).
Step 2. Re-injecting (3.14) in (3.13) gives

$$\begin{aligned}&\mathbb{E}\left[\big(\sup_{t\wedge\tau^p_{k-1}\le s\le t\wedge\tau^p_k}|X^p_s|^4\big)\mathbb{1}_{\{k\le n^*\}}\right]\\ \le\ & C^k\big(x^4+1+\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\big)\\ &+ C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)\int_0^t C^k\left(x^4+1+\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\right)e^{C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)r}dr\\ \le\ & C^k\left(x^4+1+\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\right)e^{C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)t}.\end{aligned}$$

We then get (ii) by Lipschitz property of b and σ, Cauchy-Schwartz's inequality and Itô isometry

$$\begin{aligned}
&\mathbb{E}\left[|X^p_{t\wedge\tau_k^{*-}} - X^p_{t\wedge\tau_k^{p-}}|^2 \mathbb{1}_{\{k\le n^*\}}\right] = \mathbb{E}\left[\left|\int_{t\wedge\tau_k^*}^{t\wedge\tau_k^p} [b(X_s^p)ds + \sigma(X_s^p)dW_s]\right|^2 \mathbb{1}_{\{k\le n^*\}}\right] \\
\le\ & C\mathbb{E}\left[\tau_k^p - \tau_k^*\right]\mathbb{E}\int_{t\wedge\tau_k^*}^{t\wedge\tau_k^p} |b(X_s^p)|^2\,\mathbb{1}_{\{k\le n^*\}}ds + C\mathbb{E}\int_{t\wedge\tau_k^*}^{t\wedge\tau_k^p} |\sigma(X_s^p)|^2\,\mathbb{1}_{\{k\le n^*\}}ds \\
\le\ & C\mathbb{E}\int_{t\wedge\tau_k^*}^{t\wedge\tau_k^p} \left(1+|X_s^p|^2\right)\mathbb{1}_{\{k\le n^*\}}ds\left(1+\frac{1}{\lambda p}\right) \\
\le\ & C\left(\mathbb{E}[1+(\sup_{t\wedge\tau_{k-1}^p\le s\le t\wedge\tau_k^p}|X_s^p|^4)\mathbb{1}_{\{k\le n^*\}}]\right)^{\frac{1}{2}}\left(\mathbb{E}\left|\tau_k^p-\tau_k^*\right|^2\right)^{\frac{1}{2}}\left(1+\frac{1}{\lambda p}\right) \\
\le\ & C\left(1+\frac{1}{2}C^k\left(x^4+1+\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\right)e^{C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)t}\right)\frac{1}{\lambda p}\left(1+\frac{1}{\lambda p}\right) \\
\le\ & C^{k+1}\left(x^4+1+\left(\frac{1}{(\lambda p)^4}+\frac{1}{(\lambda p)^2}+\frac{1}{\lambda p}\right)\right)e^{C^k\left(\frac{1}{(\lambda p)^3}+\frac{1}{\lambda p}+1\right)t}\left(\frac{1}{\lambda p}+\frac{1}{(\lambda p)^2}\right).
\end{aligned}$$

□

3.3.2 Estimate of $\mathcal{E}_0^{p,\eta,n}$

Recall $\mathcal{E}_0^{p,\eta,n} = \mathbb{E}\left[\tilde{v}^\eta(0,x) - Y_0^p\ |A^{*p}\right]$ (see Lemma 3.2.1), the definition of the distance $\tilde{v}^\eta(0,x) - Y_0^p$ in (3.9) and the one of event A^{*p} in (3.12). Then,

$$\mathcal{E}_0^{p,\eta,n} = \mathbb{E}\left[g(X_T^{u^*}) - g(X_T^p) + \int_0^T \left(f(X_s^{u^*}) - f(X_s^p)\right)ds + \sum_{k=1}^{n^*}\left(\kappa(X^{u^*}_{\tau_k^{*-}}) - \kappa(X^p_{\tau_k^{p-}})\right)\right]$$

as the jump dates $(\tau_k^p)_{k\ge1}$ of X^p are *well-drawn* with respect to the jump dates $(\tau_k^*)_{k\ge1}$ of X^{u^*}. By (H_Y) and Cauchy-Schwartz' inequality,

$$|\mathcal{E}_0^{p,\eta,n}|^2 \le C\underbrace{\mathbb{E}\left|X_T^{u^*} - X_T^p\right|^2}_{(i)} + C\underbrace{\mathbb{E}\int_0^T\left|X_t^{u^*} - X_t^p\right|^2 dt}_{(ii)} + Cn\underbrace{\mathbb{E}\left[\sum_{k=1}^{n^*}\left|X^{u^*}_{\tau_k^{*-}} - X^p_{\tau_k^{p-}}\right|^2\right]}_{(iii)}. \quad (3.15)$$

Proposition 3.3.1. *Expressions (i), (ii) and (iii) defined in* (3.15) *are such that:*

$$\begin{array}{rcl@{\qquad\qquad}rcl}
(i) & \le & C^1_{x,n}(\lambda,p)\frac{1}{\lambda p} & (iii) & \le & C^3_{x,n}(\lambda,p)\frac{1}{\lambda p} \\
(ii) & \le & C^2_{x,n}(\lambda,p)\frac{1}{\lambda p} & & &
\end{array}$$

where

$$\forall i\in\{1,2,3\},\quad C^i_{x,n}(\lambda,p) = \mathcal{O}_{p\to+\infty}\left(nC^ne^{C^n}\left(1+x^4\right)\right),$$

and C is a constant which does not depend either on λ, p, η or n. Injecting these 3 estimates in expression (3.15) *gives the expected bound for $\mathcal{E}_0^{p,\eta,n}$ in Proposition 3.2.1.*

The proofs of these 3 estimates use both Lemma 3.3.1 and the following result which gives a recursive rule for estimating in L^2 the difference $X^{u^*} - X^p$ at impulse dates $(\tau_k^*)_{k\ge1}$.

Lemma 3.3.2. For any $k \geq 1$ and $t \in [0,T]$:

$$\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_k} - X^p_{t\wedge\tau^{*-}_k}|^2 \mathbb{1}_{\{k\leq n^*\}}\right] \leq C^{k-1}\sum_{i=1}^{k-1}\mathbb{E}\left[|X^p_{t\wedge\tau^{*-}_i} - X^p_{t\wedge\tau^{p-}_i}|^2\mathbb{1}_{\{i\leq n^*\}}\right] + C^k\mathbb{E}\left[\int_0^{t\wedge\tau^*_k}\left|X^{u^*}_s - X^p_s\right|^2\mathbb{1}_{\{k\leq n^*\}}ds\right]$$

where $C>0$ is independent of λ, p, η and x.

Proof. Recall the definition of X^{u^*} in (3.10) and X^p in (3.11). Then, for any $k \geq 2$,

$$X^{u^*}_{\tau^{*-}_k} - X^p_{\tau^{*-}_k} = \left[X^{u^*}_{\tau^{*-}_{k-1}} - X^p_{\tau^{*-}_{k-1}}\right] + \left[\gamma(X^{u^*}_{\tau^{*-}_{k-1}}) - \gamma(X^p_{\tau^{p-}_{k-1}})\right] + \int_{\tau^*_{k-1}}^{\tau^*_k}\left[b(X^{u^*}_s) - b(X^p_s)\right]ds + \int_{\tau^*_{k-1}}^{\tau^*_k}\left[\sigma(X^{u^*}_s) - \sigma(X^p_s)\right]dW_s.$$

Let $t\in[0,T]$. Multiplying previous expression by $\mathbb{1}_{\{k\leq n^*\}}$, raising to power 2, using Jensen' inequality, Itô isometry, and Lipschitz property of b, σ and γ, we get

$$\begin{aligned}
&\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_k} - X^p_{t\wedge\tau^{*-}_k}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\\
\leq\ & C\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_{k-1}} - X^p_{t\wedge\tau^{*-}_{k-1}}|^2\mathbb{1}_{\{k\leq n^*\}}\right] + C\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_{k-1}} - X^p_{t\wedge\tau^{p-}_{k-1}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\\
&+C\mathbb{E}\int_{t\wedge\tau^*_{k-1}}^{t\wedge\tau^*_k}\left|b(X^{u^*}_s) - b(X^p_s)\right|^2\mathbb{1}_{\{k\leq n^*\}}ds + C\mathbb{E}\int_{t\wedge\tau^*_{k-1}}^{t\wedge\tau^*_k}\left|\sigma(X^{u^*}_s) - \sigma(X^p_s)\right|^2\mathbb{1}_{\{k\leq n^*\}}ds\\
\leq\ & C\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_{k-1}} - X^p_{t\wedge\tau^{*-}_{k-1}}|^2\mathbb{1}_{\{k\leq n^*\}}\right] + C\mathbb{E}\left[|X^p_{t\wedge\tau^{*-}_{k-1}} - X^p_{t\wedge\tau^{p-}_{k-1}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\\
&+C\mathbb{E}\left[\int_{t\wedge\tau^*_{k-1}}^{t\wedge\tau^*_k}\left|X^{u^*}_s - X^p_s\right|^2\mathbb{1}_{\{k\leq n^*\}}ds\right].
\end{aligned}$$

By recursion from $k-1$ to 1 together with

$$\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_1} - X^p_{t\wedge\tau^{*-}_1}|^2\mathbb{1}_{\{k\leq n^*\}}\right] \leq C\mathbb{E}\int_0^{t\wedge\tau^*_1}\left|X^{u^*}_s - X^p_s\right|^2\mathbb{1}_{\{k\leq n^*\}}ds,$$

we get

$$\begin{aligned}
&\mathbb{E}\left[|X^{u^*}_{t\wedge\tau^{*-}_k} - X^p_{t\wedge\tau^{*-}_k}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\\
\leq\ & \textstyle\sum_{i=1}^{k-1}C^{k-i}\mathbb{E}\left[|X^p_{t\wedge\tau^{*-}_i} - X^p_{t\wedge\tau^{p-}_i}|^2\mathbb{1}_{\{k\leq n^*\}}\right] + \sum_{i=1}^{k-1}C^{k-i}\mathbb{E}\int_{t\wedge\tau^*_i}^{t\wedge\tau^*_{i+1}}\left|X^{u^*}_s - X^p_s\right|^2\mathbb{1}_{\{k\leq n^*\}}ds\\
&+C^k\mathbb{E}\textstyle\int_0^{t\wedge\tau^*_1}\left|X^{u^*}_s - X^p_s\right|^2\mathbb{1}_{\{k\leq n^*\}}ds\\
\leq\ & C^{k-1}\textstyle\sum_{i=1}^{k-1}\mathbb{E}\left[|X^p_{t\wedge\tau^{*-}_i} - X^p_{t\wedge\tau^{p-}_i}|^2\mathbb{1}_{\{i\leq n^*\}}\right] + C^k\mathbb{E}\int_0^{t\wedge\tau^*_k}\left|X^{u^*}_s - X^p_s\right|^2\mathbb{1}_{\{k\leq n^*\}}ds.
\end{aligned}$$

□

Proof of Proposition 3.3.1 (ii). Let us show that

$$\mathbb{E}\int_0^T \left|X_t^{u^*} - X_t^p\right|^2 dt \leq Ce^{C^n}\left(C(n+1)\frac{1}{\lambda p} + \beta_n(\lambda, p)\right)$$

where

$$\beta_n(\lambda,p) \quad = \quad nC^n A_{x,n}(\lambda,p)\left(\frac{1}{\lambda p} + \frac{1}{(\lambda p)^2}\right). \tag{3.16}$$

For any $t \in [0,T]$, define

$$n_t = \max\{k \geq 0, \tau_k^* < t\} \leq n^* \leq n \text{ a.s.}$$

For any $t \in [\tau_1^*, T]$ ($n_t \geq 1$), we have

$$\begin{aligned} X_t^{u^*} - X_t^p = &\left[X^{u^*}_{\tau^{*-}_{n_t}} - X^p_{\tau^{*-}_{n_t}}\right] + \left[\gamma(X^{u^*}_{\tau^{*-}_{n_t}}) - \gamma(X^p_{\tau^{p-}_{n_t}})\mathbb{1}_{\left\{\tau^p_{n_t} \leq t\right\}}\right] \\ &+ \int_{\tau^*_{n_t}}^t \left[b(X_s^{u^*}) - b(X_s^p)\right] ds + \int_{\tau^*_{n_t}}^t \left[\sigma(X_s^{u^*}) - \sigma(X_s^p)\right] dW_s. \end{aligned}$$

By (H_X) and Lemma 3.3.2 with $t = T$ and $k = n_t$, we get

$$\begin{aligned} &\mathbb{E}\left|X_t^{u^*} - X_t^p\right|^2 \\ \leq\ & C\mathbb{E}\left|X^{u^*}_{\tau^{*-}_{n_t}} - X^p_{\tau^{*-}_{n_t}}\right|^2 + C\mathbb{E}\left|X^p_{\tau^{*-}_{n_t}} - X^p_{\tau^{p-}_{n_t}}\right|^2 + C\mathbb{E}\left|\gamma(X^p_{\tau^{p-}_{n_t}})\mathbb{1}_{\left\{\tau^p_{n_t} > t\right\}}\right|^2 + C\mathbb{E}\int_{\tau^*_{n_t}}^t \left|X_s^{u^*} - X_s^p\right|^2 ds \\ \leq\ & C^n \textstyle\sum_{i=1}^n \mathbb{E}\left[|X^p_{\tau_i^{*-}} - X^p_{\tau_i^{p-}}|^2 \mathbb{1}_{\{i \leq n^*\}}\right] + C\mathbb{P}\left[\tau^p_{n_t} > t\right] + C^n\mathbb{E}\int_0^t \left|X_s^{u^*} - X_s^p\right|^2 ds. \end{aligned}$$

Together with Lemma 3.3.1-(ii), we get, $t \in [\tau_1^*, T]$,

$$\mathbb{E}\left|X_t^{u^*} - X_t^p\right|^2 \leq nC^n A_{x,n}(\lambda,p)\left(\frac{1}{\lambda p} + \frac{1}{(\lambda p)^2}\right) + C\mathbb{P}\left[\tau^p_{n_t} > t\right] + C^n\mathbb{E}\int_0^t \left|X_s^{u^*} - X_s^p\right|^2 ds. \tag{3.17}$$

With Lipschitz property of b and σ, we have also for any $t \in [0, \tau_1^*)$

$$\mathbb{E}\left|X_t^{u^*} - X_t^p\right|^2 \leq C\mathbb{E}\int_0^t \left|X_s^{u^*} - X_s^p\right|^2 ds. \tag{3.18}$$

Let $r \in [0,T]$. (3.17) and (3.18) leads to

$$\begin{aligned} \mathbb{E}\int_0^r \left|X_t^{u^*} - X_t^p\right|^2 dt \leq\ & nC^n A_{x,n}(\lambda,p)\left(\frac{1}{\lambda p} + \frac{1}{(\lambda p)^2}\right) + C\int_{\tau_1^*}^T \mathbb{P}\left[\tau^p_{n_t} > t\right] dt \\ &+\ C^n \int_0^r \left(\mathbb{E}\int_0^t \left|X_s^{u^*} - X_s^p\right|^2 ds\right) dt. \end{aligned}$$

Recalling $(\tau_k^p)_{k\geq 1} \in A^{*p}$, we have

$$\mathbb{P}\left[\tau^p_{n_t} > t | \tau^*_{n_t}\right] = \mathbb{P}\left[\tau^p_{n_t} - \tau^*_{n_t} > t - \tau^*_{n_t} | \tau^*_{n_t}\right] = e^{-\lambda p(t - \tau^*_{n_t})}.$$

Thus, the second term of the right-hand side in previous inequality, is such that

$$\begin{aligned}\int_{\tau_1^*}^T \mathbb{P}\left[\tau_{n_t}^p > t\right] dt &= \int_{\tau_1^*}^T \mathbb{E}\left[\mathbb{P}\left[\tau_{n_t}^p > t|\tau_{n_t}^*\right]\right] dt \leq \mathbb{E}\left[\int_0^T e^{-\lambda p(t-\tau_{n_t}^*)} dt\right] \\ &= \mathbb{E}\left[\sum_{i=0}^{n^*-1} \int_{\tau_i^*}^{\tau_{i+1}^*} e^{-\lambda p(t-\tau_i^*)} dt + \int_{\tau_{n^*}^*}^T e^{-\lambda p(t-\tau_{n^*}^*)} dt\right] \\ &= \mathbb{E}\left[\sum_{i=0}^{n^*-1} \frac{1}{\lambda p}\left(1 - e^{-\lambda p(\tau_{i+1}^* - \tau_i^*)}\right) + \frac{1}{\lambda p}\left(1 - e^{-\lambda p(T-\tau_{n^*}^*)}\right)\right] \\ &\leq (n+1)\frac{1}{\lambda p}.\end{aligned}$$

Together with (3.16), this leads to

$$\mathbb{E}\int_0^r \left|X_t^{u^*} - X_t^p\right|^2 dt \leq C(n+1)\frac{1}{\lambda p} + \beta_n(\lambda, p) + C^n \int_0^r \left(\mathbb{E}\int_0^t \left|X_s^{u^*} - X_s^p\right|^2 ds\right) dt, \quad \forall r \in [0, T].$$

Now apply Gronwall's lemma to $r \to \mathbb{E}\int_0^r \left|X_t^{u^*} - X_t^p\right|^2 dt$ to obtain

$$\mathbb{E}\int_0^r \left|X_t^{u^*} - X_t^p\right|^2 dt \leq e^{C^n r}\left(C(n+1)\frac{1}{\lambda p} + \beta_n(\lambda, p)\right), \quad \forall r \in [0, T],$$

which leads to expected result. □

Proof of Proposition 3.3.1 (i). We have

$$\begin{aligned}X_T^{u^*} - X_T^p = &\left[X_{\tau_{n^*}^{*-}}^{u^*} - X_{\tau_{n^*}^{*-}}^p\right] + \left[\gamma(X_{\tau_{n^*}^{*-}}^{u^*}) - \gamma(X_{\tau_{n^*}^{p-}}^p)\mathbb{1}_{\{\tau_{n^*}^p \leq T\}}\right] \\ &+ \int_{\tau_{n^*}^*}^T \left[b(X_s^{u^*}) - b(X_s^p)\right] ds + \int_{\tau_{n^*}^*}^T \left[\sigma(X_s^{u^*}) - \sigma(X_s^p)\right] dW_s.\end{aligned}$$

In the same way as in the proof for part (ii), by (H_X) and Lemma 3.3.2 with $t = T$ and $k = n^*$, we get;

$$\begin{aligned}&\mathbb{E}\left|X_T^{u^*} - X_T^p\right|^2 \\ \leq\ & C\mathbb{E}\left|X_{\tau_{n^*}^{*-}}^{u^*} - X_{\tau_{n^*}^{*-}}^p\right|^2 + C\mathbb{E}|X_{\tau_{n^*}^{*-}}^p - X_{\tau_{n^*}^{p-}}^p|^2 + C\mathbb{E}\int_{\tau_{n^*}^*}^T \left|X_s^{u^*} - X_s^p\right|^2 ds \\ \leq\ & C^n \sum_{i=1}^n \mathbb{E}\left[|X_{\tau_i^{*-}}^p - X_{\tau_i^{p-}}^p|^2 \mathbb{1}_{\{i \leq n^*\}}\right] + C^n\mathbb{E}\int_0^T \left|X_s^{u^*} - X_s^p\right|^2 ds.\end{aligned}$$

Using Lemma 3.3.1-(ii) and the bound for part (ii),

$$\mathbb{E}\left|X_T^{u^*} - X_T^p\right|^2 \leq nC^n A_{x,n}(\lambda, p)\left(\frac{1}{\lambda p} + \frac{1}{(\lambda p)^2}\right) + C^n e^{C^n}\left(C(n+1)\frac{1}{\lambda p} + \beta_n(\lambda, p)\right).$$

in which $\beta_n(\lambda, p)$ is defined in (3.16). □

Proof of Proposition 3.3.1 (iii) of the error. Let us show that

$$\mathbb{E}\left[\sum_{k=1}^{n^*} \left|\kappa(X_{\tau_k^{*-}}^{u^*}) - \kappa(X_{\tau_k^{p-}}^p)\right|^2\right] \leq CnA_{x,n}(\lambda, p)\left(\frac{1}{\lambda p} + \frac{1}{(\lambda p)^2}\right) + Cne^{C^n}\beta_{n-1}(\lambda, p)$$

in which $\beta_n(\lambda,p)$ is defined in (3.16). With Lipschitz property of κ,

$$\begin{aligned}&\mathbb{E}\left[\sum_{k=1}^{n^*}\left|\kappa(X^{u^*}_{\tau_k^{*-}})-\kappa(X^{p}_{\tau_k^{p-}})\right|^2\right]\\ \leq\ & C\sum_{k=1}^{n}\mathbb{E}\left[|X^{p}_{\tau_k^{*-}}-X^{p}_{\tau_k^{p-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]+C\sum_{k=1}^{n}\mathbb{E}\left[|X^{u^*}_{\tau_k^{*-}}-X^{p}_{\tau_k^{*-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right].\end{aligned}$$

An estimate of the first term is given by Lemma 3.3.1-(ii):

$$\mathbb{E}\left[|X^{p}_{\tau_k^{*-}}-X^{p}_{\tau_k^{p-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\leq CA_{x,n}(\lambda,p)\left(\frac{1}{\lambda p}+\frac{1}{(\lambda p)^2}\right),\quad \forall k=1,\dots,n.$$

By Lemma 3.3.2 and Lemma 3.3.1-(ii), the second term is such that, for any $t\in[0,T]$,

$$\begin{aligned}&\mathbb{E}\left[|X^{u^*}_{t\wedge\tau_k^{*-}}-X^{p}_{t\wedge\tau_k^{*-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\\ \leq\ & C^{k-1}\sum_{i=1}^{k-1}\mathbb{E}\left[|X^{p}_{t\wedge\tau_i^{*-}}-X^{p}_{t\wedge\tau_i^{p-}}|^2\mathbb{1}_{\{i\leq n^*\}}\right]+C^k\mathbb{E}\int_0^t\left|X^{u^*}_{s\wedge\tau_k^{*}}-X^{p}_{s\wedge\tau_k^{*}}\right|^2\mathbb{1}_{\{k\leq n^*\}}ds\\ \leq\ & \underbrace{(k-1)C^{k-1}A_{x,k-1}(\lambda,p)\left(\frac{1}{\lambda p}+\frac{1}{(\lambda p)^2}\right)}_{=\,\beta_{k-1}(\lambda,p)}+C^k\int_0^t\mathbb{E}\left[\left|X^{u^*}_{s\wedge\tau_k^{*}}-X^{p}_{s\wedge\tau_k^{*}}\right|^2\mathbb{1}_{\{k\leq n^*\}}\right]ds.\end{aligned}$$

Applying Gronwall's lemma to $t\to\mathbb{E}\left[|X^{u^*}_{t\wedge\tau_k^{*-}}-X^{p}_{t\wedge\tau_k^{*-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]$, we get

$$\mathbb{E}\left[|X^{u^*}_{t\wedge\tau_k^{*-}}-X^{p}_{t\wedge\tau_k^{*-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\ \leq\ e^{C^nt}\beta_{n-1}(\lambda,p),\quad\forall k=1,\dots,n.$$

We get an estimate for $\mathbb{E}\left[|X^{u^*}_{\tau_k^{*-}}-X^{p}_{\tau_k^{*-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]$ when taking $t=T$ in previous expression. □

3.3.3 Estimate of $\hat{\mathcal{E}}_0^{p,\eta,n}$

The estimate for $\hat{\mathcal{E}}_0^{p,\eta,n}$ in Proposition 3.2.1 is straightforward with the same reasoning as in the proof for estimating $\mathcal{E}_0^{p,\eta,n}$ (cf. Proposition 3.3.1). Even if we are not on event A^{*p}, it is enough to use wider estimates to get a bound with order 0 as $p\to+\infty$. You just need to repeat step by step the ideas of previous Paragraph 3.3.2 using many times the following assumption on γ in (H_X)

$$\forall x,\ |\gamma(x)|\leq C.$$

It allows in particular to derive a recursive rule as the one in Lemma 3.3.2: for any $k\geq 1$ and $t\in[0,T]$,

$$\mathbb{E}\left[|X^{u^*}_{t\wedge\tau_k^{*-}}-X^{p}_{t\wedge\tau_k^{*-}}|^2\mathbb{1}_{\{k\leq n^*\}}\right]\leq(k-1)C^{k-1}+C^k\mathbb{E}\int_0^{t\wedge\tau_k^*}\left|X^{u^*}_{s}-X^{p}_{s}\right|^2\mathbb{1}_{\{k\leq n^*\}}ds.\quad(3.19)$$

This finally leads to

$$\left|\hat{\mathcal{E}}_0^{p,\eta,n}\right|^2\leq C\left(nC^ne^{C^n}\right),$$

in which $C>0$ is a constant which does not depend either on λ, p, η or n.

Chapter 4

Estimation of the discretization error

In this chapter, we fix a penalization parameter $p > 0$ and study the error made when solving in discrete time previously introduced penalized BSDE with jumps for solving impulse control problem (1.2). The decoupled Forward Backward SDE with jumps (2.6)-(2.12) that we consider can be rewritten as

$$X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s + \int_0^t \gamma(X_{s^-})dN_s, \quad 0 \leq t \leq T \tag{4.1}$$

$$Y_t^p = g(X_T) + \int_t^T f^p(X_s, V_s^p)ds - \int_t^T Z_s^p dW_s - \int_t^T V_s^p d\tilde{N}_s, \quad \forall 0 \leq t \leq T \tag{4.2}$$

in which $\tilde{N}$ is the compensated Poisson process associated to Poisson N with jump intensity λ, i.e. $d\tilde{N}_t = dN_t - \lambda dt$ and the driver of the penalized BSDE is denoted by f^p:

$$\forall (x, v) \in \mathbb{R}^d \times \mathbb{R}, \quad f^p(x, v) := f(x) + \big(p(v + \kappa(x))^+ - v\big)\,\lambda. \tag{4.3}$$

In Section 4.1, we first present the considered discrete-time approximation to FBSDE (4.1)-(4.2), which is a particular case of the approximation introduced by Bouchard and Elie [19]. We then discuss the main results of this chapter: that is, the convergence rate of the discretization error and its behavior with respect to the jump intensity λ, the penalization parameter p and the time step of the discretization grid. Besides, we again justify our approach by penalization, which seems to be the only one relevant in our framework for solving problem (1.2) when using BSDEs. In particular, one cannot use a *scheme by projection* to get rid of the constraint on jumps in (2.5), see Remark 4.1.3.

Sections 4.2 and 4.3 are dedicated to the computation of the bounds on the discretization error. A global convergence rate of our approximation by penalization for solving problem (1.2) is provided in Theorem 4.4.1. This will be useful for practical applications of the method, see Chapter 5.

4.1 Discrete-time approximation

Let us introduce an integer $N \in \mathbb{N}^*$, a regular time grid

$$\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}$$

and set $|\pi| := \Delta t_{n+1} = t_{n+1} - t_n$, $\forall n < N$. The solution X to (4.1) is approximated on π by its Euler scheme X^π, namely:

$$\begin{cases} X_0^\pi = x \\ X_{t_{n+1}}^\pi = X_{t_n}^\pi + b(X_{t_n}^\pi)\Delta t_{n+1} + \sigma(X_{t_n}^\pi)\Delta W_{t_{n+1}} + \gamma(X_{t_n}^\pi)\Delta N_{t_{n+1}}, \forall t_n \in \pi, t_n > 0 \end{cases} \tag{4.4}$$

where $\Delta W_{t_{n+1}} = W_{t_{n+1}} - W_{t_n}$ and $\Delta N_{t_{n+1}} = N_{t_{n+1}} - N_{t_n}$. One can then approximate the solution (Y^p, Z^p, V^p) to (4.2) by the discrete-time process $(\bar{Y}_{t_n}^{p,\pi}, \bar{Z}_{t_n}^{p,\pi}, \bar{V}_{t_n}^{p,\pi})_{n=0,\dots,N}$ defined backward in time by

$$\begin{cases} \bar{Y}_{t_N}^{p,\pi} = g(X_{t_N}^\pi) \\ \forall t_n \in \pi, t_n < T, \forall t \in [t_n, t_{n+1}) : \\ \qquad \bar{V}_t^{p,\pi} = \frac{1}{\lambda \Delta t_{n+1}} \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi} \Delta \tilde{N}_{t_{n+1}}\right] \\ \qquad \bar{Z}_t^{p,\pi} = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi} \Delta W_{t_{n+1}}\right] \\ \qquad \bar{Y}_t^{p,\pi} = \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi}\right] + f^p(X_{t_n}^\pi, \bar{V}_{t_n}^{p,\pi})\Delta t_{n+1} \end{cases} \tag{4.5}$$

where $\mathbb{E}_{t_n}[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_{t_n}]$ and $\Delta\tilde{N}$ is the compensated version of the Poisson increment ΔN. The scheme (4.5) is written piecewise constant on $[0,T]$ for sake of simplicity in further notations.

Remark 4.1.1 (Intuition for the backward numerical scheme (4.5))**.** The idea (cf. Elie [48] or Bouchard and Touzi [20] in the no-jump case) to write the backward scheme in discrete time, is to replace (X, V^p) in the driver f^p of BSDE (4.2), by $\mathcal{F}_{t_n}$-measurable random variables $(\tilde{X}_{t_n}, \tilde{V}_{t_n}^p)$. We naturally take $\tilde{X}_{t_n} = X_{t_n}^\pi$, see (4.4). Writing the BSDE (4.2) on $[t_n, t_{n+1})$ and taking conditional expectation $\mathbb{E}_{t_n}[\cdot]$, we get the approximation

$$Y_{t_n}^p \approx \mathbb{E}_{t_n}\left[Y_{t_{n+1}}^p\right] + f^p(X_{t_n}^\pi, \tilde{V}_{t_n}^p)\lambda\Delta t_{n+1},$$

so that we obtain the following backward scheme for the numerical approximation of Y^p on π, denoted $\bar{Y}^{p,\pi}$

$$\bar{Y}_{t_n}^{p,\pi} = \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi}\right] + f^p(X_{t_n}^\pi, \tilde{V}_{t_n}^p)\lambda\Delta t_{n+1}.$$

It remains to choose $\tilde{V}_{t_n}^p$ in terms of $\bar{Y}_{t_{n+1}}^{p,\pi}$. By the representation theorem applied to $\bar{Y}_{t_n}^{p,\pi}$ on $[t_n, t_{n+1})$, there exists a unique couple of processes $(Z^{p,\pi}, V^{p,\pi}) \in \mathrm{L}^2(W) \times \mathrm{L}^2(N)$ such that

$$\bar{Y}_{t_{n+1}}^{p,\pi} - \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi}\right] = \int_{t_n}^{t_{n+1}} Z_s^{p,\pi} dW_s + \int_{t_n}^{t_{n+1}} V_s^{p,\pi} d\tilde{N}_s. \tag{4.6}$$

Define $\bar{Z}^{p,\pi}$ (resp. $\bar{V}^{p,\pi}$) the numerical approximation of Z^p (resp. V^p) on π by

$$\bar{Z}_{t_n}^{p,\pi} = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi} \Delta W_{t_{n+1}}\right], \qquad \bar{V}_{t_n}^{p,\pi} = \frac{1}{\lambda \Delta t_{n+1}} \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi} \Delta \tilde{N}_{t_{n+1}}\right]. \tag{4.7}$$

Thanks to (4.6), we get its expression in term of $Z^{p,\pi}$ (resp. $V^{p,\pi}$)

$$\bar{Z}_{t_n}^{p,\pi} = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} Z_s^{p,\pi} ds\right], \qquad \bar{V}_{t_n}^{p,\pi} = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} V_s^{p,\pi} ds\right],$$

which can be interpreted as follows: $\bar{Z}_{t_n}^{p,\pi}$ (resp. $\bar{V}_{t_n}^{p,\pi}$) is the best $\mathrm{L}^2_{[t_n,t_{n+1}]}(W)$-approximation of the process $(Z_s^{p,\pi})_{s\in[t_n,t_{n+1})}$ (resp. $(V_s^{p,\pi})_{s\in[t_n,t_{n+1})}$) by a $\mathcal{F}_{t_n}$-measurable random variable (viewed as constant processes on $[t_n, t_{n+1})$). Taking $\tilde{V}_{t_n}^p = \bar{V}_{t_n}^{p,\pi}$, we get

$$\bar{Y}_{t_n}^{p,\pi} = \mathbb{E}_{t_n}\left[\bar{Y}_{t_{n+1}}^{p,\pi}\right] + f^p(X_{t_n}^\pi, \bar{V}_{t_n}^{p,\pi})\lambda\Delta t_{n+1}, \tag{4.8}$$

which gives, together with (4.6),

$$\bar{Y}^{p,\pi}_{t_n} = \bar{Y}^{p,\pi}_{t_{n+1}} + f^p(X^\pi_{t_n}, \bar{V}^{p,\pi}_{t_n})\lambda\Delta t_{n+1} - \int_{t_n}^{t_{n+1}} Z^{p,\pi}_s dW_s - \int_{t_n}^{t_{n+1}} V^{p,\pi}_s d\tilde{N}_s.$$

Finally, we define $Y^{p,\pi}$ as the extended version of $\bar{Y}^{p,\pi}$ on $[0,T]$

$$\begin{aligned}\forall t \in [t_n, t_{n+1}), Y^{p,\pi}_t &= \bar{Y}^{p,\pi}_{t_n} - f^p(X^\pi_{t_n}, \bar{V}^{p,\pi}_{t_n})\lambda(t-t_n) + \int_{t_n}^{t} Z^{p,\pi}_s dW_s + \int_{t_n}^{t} V^{p,\pi}_s d\tilde{N}_s \\ &= \bar{Y}^{p,\pi}_{t_{n+1}} + f^p(X^\pi_{t_n}, \bar{V}^{p,\pi}_{t_n})\lambda(t_{n+1}-t) - \int_{t}^{t_{n+1}} V^{p,\pi}_s dW_s - \int_{t}^{t_{n+1}} V^{p,\pi}_s d\tilde{N}_s\end{aligned}$$

to obtain

$$\bar{Y}^{p,\pi}_{t_n} = Y^{p,\pi}_{t_n} = \frac{1}{\Delta t_{n+1}}\mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} Y^{p,\pi}_s ds\right],$$

meaning that $\bar{Y}^{p,\pi}_{t_n}$ is the best $\mathrm{L}^2_{[t_n,t_{n+1}]}(W)$-approximation of the process $(Y^{p,\pi}_s)_{s\in[t_n,t_{n+1})}$ by a $\mathcal{F}_{t_n}$-measurable random variable (viewed as a constant process on $[t_n, t_{n+1})$). Equations (4.7) and (4.8) lead to the backward scheme in (4.5).

Let us consider the classical discretization error between the continuous-time solution (Y^p, Z^p, V^p) in (4.2) and its discrete time approximation $(\bar{Y}^{p,\pi}, \bar{Z}^{p,\pi}, \bar{V}^{p,\pi})$ in (4.5), that is

$$\mathcal{E}^\pi := \mathcal{E}^\pi(Y^p) + \mathcal{E}^\pi(Z^p) + \mathcal{E}^\pi(V^p),$$

where

$$\begin{aligned}|\mathcal{E}^\pi(Y^p)|^2 &:= \sup_{0\le t\le T}\mathbb{E}\left|Y^p_t - \bar{Y}^{p,\pi}_t\right|^2 = \max_{n=0,\dots,N-1}\sup_{t_n\le t\le t_{n+1}}\mathbb{E}\left|Y^p_t - \bar{Y}^{p,\pi}_{t_n}\right|^2, \\ |\mathcal{E}^\pi(Z^p)|^2 &:= \left\|Z^p - \bar{Z}^{p,\pi}\right\|^2_{\mathrm{L}^2(W)} = \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\mathbb{E}\left|Z^p_t - \bar{Z}^{p,\pi}_{t_n}\right|^2 dt, \\ |\mathcal{E}^\pi(V^p)|^2 &:= \left\|V^p - \bar{V}^{p,\pi}\right\|^2_{\mathrm{L}^2(N)} = \sum_{n=0}^{N-1}\int_{t_n}^{t_{n+1}}\mathbb{E}\left|V^p_t - \bar{V}^{p,\pi}_{t_n}\right|^2 \lambda dt.\end{aligned}$$

Our main interest consists in providing *explicit upper bounds* on the discretization error $\mathcal{E}^\pi(Y^p)$, $\mathcal{E}^\pi(Z^p)$ and $\mathcal{E}^\pi(V^p)$ in terms on approximation parameters λ, p and $|\pi|$. The estimate computations performed in this chapter follow from the same arguments as Elie [48], but were rigorously carried out with respect to the dependence in parameters λ and p: these parameters are in particular involved in the estimations because the Lipschitz coefficient of the BSDE driver f^p depends on λ and p, see (4.3).

Such an error estimation is new, to our best knowledge, and is essential for numerical purposes: this allows a better understanding of the impact of approximation parameters (λ, p) on the error made when solving FBSDE (4.1)-(4.2) in discrete time. As a by-product, it allows to adjust, in practical experiments, the fineness of the time grid π in relation to (λ, p).

The impact of the penalization coefficient p on the convergence of backward discrete-time schemes is well-know in practice, even if we did not find any explicit computation in the literature (see for example the numerical experiments of Lemor [80] for the resolution by penalization of a BSDE with one reflecting barrier). Basically, as p increases at fixed discrete-time step, the

quantity $f^p(\cdot)\Delta t_{n+1}$ explodes, leading to a numerical explosion of the approximate values $\bar{Y}^p$, see (4.5). On the other hand, in Elie [48], the estimation of the discretization error of a BSDE with jumps by the backward scheme presented above (leading to a $|\pi|^{\frac{1}{2}}$ convergence rate) is performed without taking into account the dependence in the intensity λ. And yet, the author alludes to its critical role in practice (see [48] page 146).

Let us underline that the main difficulty comes from the estimation of residual errors involving the path-regularity of the continuous-time solution (Y^p, Z^p, V^p) that naturally appear when estimating $\mathcal{E}^\pi$. We handle with them by classical regularization and Malliavin differentiation arguments. For a presentation on Malliavin calculus for backward stochastic differential equations with jumps, the reader may refer for example to [48]. Let us consider the assumption

(H′) The maps b, σ, γ, and g belong to $\mathcal{C}_b^1(\mathbb{R}^d)$[1] and have Lipschitz continuous derivatives.

The convergence rate of discretization error $\mathcal{E}^\pi$ would be of order $|\pi|^{\frac{1}{2}}$, see Elie [48], if the considered BSDE (4.2) had a driver belonging to $\mathcal{C}_b^1$. Here, it is not the case due to our penalization approach, see (4.3). Under an alternative assumption to (H′), which concerns the invertibility of $\nabla\gamma + I_d$, this convergence rate still holds (see in [48] as well). However, the key argument for this (see Proposition 1.4.5 page 120 in [48]) does not allow to provide an *explicit* (with respect to p) bound on the regularity error of Z^p, since it is based on representation theorems.

We will thus use assumption (H′) for estimating regularity errors of solution (Y^p, Z^p, V^p). Because of the lack of first order regularity of the BSDE driver f^p, see (4.3), classical regularization arguments for the FBSDE coefficients allow us to provide an explicit convergence rate of order $|\pi|^{\frac{1}{2}}$ for error $\mathcal{E}^\pi(Y^p)$ and $\mathcal{E}^\pi(V^p)$, but only of order $|\pi|^{\frac{1}{4}}$ for error $\mathcal{E}^\pi(Z^p)$, see Proposition 4.3.3 and Corollary 4.4.1.

We show (cf. Proposition 4.2.1) that a necessary condition to those convergence rates is

$$|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right).$$

Besides, the error $\mathcal{E}^\pi$ is shown to exponentially grow with (λp^2), see Corollary 4.4.1. This is due to the linear dependence in λ and p of the BSDE driver f^p, see (4.3), and estimate computations based on the use of Gronwall's lemma. For a fixed $|\pi|$, the convergence rate would thus strongly deteriorate as λ or p increase.

Remark 4.1.2. It is clear from the computations that are performed in the proofs that all the results of this chapter hold if we let the maps b, σ, γ, f, κ and g depend on t, as soon as these functions are $\frac{1}{2}$-Hölder in t and the assumptions (H) and (H′) are satisfied uniformly in t.

Remark 4.1.3 (A justification of the penalization approach)**.** When a decoupled FBSDE involves a constraint on the component Y, one can prove the convergence of the numerical backward scheme by projection of the constraint on Y. We refer for example to Bouchard and Chassagneux [18] for BSDEs with reflecting barriers of normal type and to Chassagneux et al. [32] for oblique reflections. However, the same convergence arguments do not apply to BSDEs with jumps involving a constraint on the jump component V, since no minimality condition for component K is available. In the following, we show why, in our particular case, a numerical scheme in

[1] A map is said to belong to $\mathcal{C}_b^1$ if it is in $\mathcal{C}^1$ with bound derivatives and to belong to $\mathcal{C}_b^2$ if it is in $\mathcal{C}^2$ with bound derivatives up to second order derivatives.

which the constraint on V would be projected is irrelevant to approximate the minimal solution (Y, Z, V, K) to BSDE with constrained jumps (2.5). This justification remains true for BSDEs with constrained jump as the one in (2.5) when

- neither the integral gain f nor the intervention gain κ depend on the v-variable,
- the constraint on jumps has the form $-V \geq \kappa(\cdot)$.

Denote by (Y^π, Z^π, V^π) the approximation to (Y, Z, V) in (2.5), when using a naive projection of the constraint on V at each discrete time $t_n \in \pi$, that is:

$$\begin{cases} Y^\pi_{t_N} = g(X^\pi_{t_N}) \\ \forall t_n \in \pi, t_n < T : \\ \qquad \tilde{V}^\pi_{t_n} = \frac{1}{\lambda \Delta t_{n+1}} \mathbb{E}_{t_n}\left[Y^\pi_{t_{n+1}} \Delta \tilde{N}_{t_{n+1}} \right] \\ \qquad V^\pi_{t_n} = \min\left\{ \tilde{V}^\pi_{t_n}; -\kappa(X^\pi_{t_n}) \right\} \\ \qquad Z^\pi_{t_n} = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[Y^\pi_{t_{n+1}} \Delta W_{t_{n+1}} \right] \\ \qquad Y^\pi_{t_n} = \mathbb{E}_{t_n}\left[Y^\pi_{t_{n+1}} \right] + f(X^\pi_{t_n}) \Delta t_{n+1} - V^\pi_{t_n} \lambda \Delta t_{n+1}. \end{cases}$$

Comparing this scheme to backward scheme (4.5) for the associated penalized BSDE with jumps (4.2), we have

$$\begin{array}{c} \forall p > 0, \quad \bar{Y}^{p,\pi}_{t_N} = Y^\pi_{t_N} = g(X^\pi_{t_N}), \\ \forall p > 0, \forall n < N, \quad \bar{Y}^{p,\pi}_{t_{n+1}} = Y^\pi_{t_{n+1}} \Longrightarrow \bar{V}^{p,\pi}_{t_n} = \tilde{V}^\pi_{t_n} \text{ and } \bar{Z}^{p,\pi}_{t_n} = Z^\pi_{t_n}. \end{array}$$

Taking $p = 1$ in backward scheme (4.5), notice that for any $n < N$,

$$\begin{array}{rrcl} & \bar{Y}^{1,\pi}_{t_n} & = & \mathbb{E}_{t_n}\left[\bar{Y}^{1,\pi}_{t_{n+1}} \right] + f(X^\pi_{t_n}) + \left[(\bar{V}^{1,\pi}_{t_n} + \kappa(X^\pi_{t_n}))^+ - \bar{V}^{1,\pi}_{t_n} \right] \lambda \Delta t_{n+1}, \\ \text{with} & Y^\pi_{t_n} & = & \mathbb{E}_{t_n}\left[Y^\pi_{t_{n+1}} \right] + f(X^\pi_{t_n}) \Delta t_{n+1} + \left((\tilde{V}^\pi_{t_n} + \kappa(X^\pi_{t_n}))^- - \kappa(X^\pi_{t_n}) \right) \lambda \Delta t_{n+1} \\ & & = & \mathbb{E}_{t_n}\left[Y^\pi_{t_{n+1}} \right] + f(X^\pi_{t_n}) \Delta t_{n+1} + \left[(\tilde{V}^\pi_{t_n} + \kappa(X^\pi_{t_n}))^+ - \tilde{V}^\pi_{t_n} \right] \lambda \Delta t_{n+1}, \end{array}$$

so that the two schemes are equivalent on π in the sense that

$$\begin{array}{c} \forall n \leq N, \quad \bar{Y}^{1,\pi}_{t_n} = Y^\pi_{t_n}, \\ \forall n < N, \quad \bar{Z}^{1,\pi}_{t_n} = Z^\pi_{t_n}, \\ \forall n < N, \quad \bar{V}^{1,\pi}_{t_n} = \tilde{V}^\pi_{t_n}. \end{array}$$

And yet under (H) and (H′), the numerical approximation (4.5) converges to the solution to (4.2), see [48], namely

$$(\bar{Y}^{1,\pi}, \bar{Z}^{1,\pi}, \bar{V}^{1,\pi}) \longrightarrow (Y^1, Z^1, V^1) \quad \text{as } |\pi| \to 0,$$

where (Y^1, Z^1, V^1) is the solution to (4.2) when the penalization coefficient is equal to 1. Together with the monotone convergence of $(Y^p)_{p>0}$ in Proposition 2.2.2

$$Y^p \nearrow Y \quad \text{as } p \to +\infty,$$

this proves that this naive projected backward scheme for Y^π cannot converge to the value process Y of the minimal solution to BSDE with constrained jumps (2.5).

Notation Throughout this chapter, C will denote a strictly positive constant (which may change from line to line in the estimations) depending only on Lipschitz coefficients or bounds implied by conditions (H) and (H′) and constants T, $|b(0)|$, $|\sigma(0)|$, $|\gamma(0)|$, $|f(0,0)|$, $|\kappa(0)|$, $|g(0)|$. In particular, such a constant C does not depend either on x, λ, p, $|\pi|$ or any additional approximation parameter which may be introduced hereafter.

4.2 A first estimate of the error due to discretization

In Lemma 4.2.1, we recall a useful a priori estimate for jump diffusion X solution to (4.1) and in Lemma 4.2.2, the well-known estimation of the distance between X and its Euler approximation (4.4), see for example Bouchard and Elie [19].

Lemma 4.2.1. Assume (H_X). Then

$$\|X\|_{\mathcal{S}^2}^2 := \mathbb{E}\left[\sup_{0\le t\le T}|X_t|^2\right] \le \bar{C}_{x,\lambda}^{2,e}, \tag{4.9}$$

where[2]

$$\bar{C}_{x,\lambda}^{2,e} = C(1+x^2)(1+\lambda)^2 e^{C(1+\lambda)^2}. \tag{4.10}$$

Proof. Straightforward using Jensen's and Doob's inequalities and Gronwall's lemma. Indeed, with assumption (H_X), we have $\forall t \in [0,T]$,

$$\begin{aligned}
& \mathbb{E}\left[\sup_{0\le s\le t}|X_s|^2\right] \\
\le\ & 4x^2 + 4T\mathbb{E}\left[\int_0^t |b(X_s)+\gamma(X_s)\lambda|^2\,ds\right] + 4\mathbb{E}\left[\int_0^t |\sigma(X_s)|^2\,ds\right] + 4\mathbb{E}\left[\int_0^t |\gamma(X_s)|^2\,\lambda ds\right] \\
\le\ & Cx^2 + C\left(1+\lambda+\lambda^2\right)\int_0^t \left(1+\mathbb{E}|X_s|^2\right) ds \\
\le\ & C\left(x^2+1+\lambda+\lambda^2\right) + C\left(1+\lambda+\lambda^2\right)\int_0^t \mathbb{E}\left[\sup_{0\le u\le s}|X_u|^2\right] ds
\end{aligned}$$

and the result is obtained by applying Gronwall's lemma to $t \mapsto \mathbb{E}\left[\sup_{0\le s\le t}|X_s^p|^2\right]$, leading to

$$\mathbb{E}\left[\sup_{0\le s\le t}|X_s|^2\right] \le C\left(x^2+1+\lambda+\lambda^2\right)e^{C(1+\lambda+\lambda^2)t} \le C(1+x^2)(1+\lambda)^2 e^{C(1+\lambda)^2}.$$

□

Lemma 4.2.2. Assume (H_X). Then, the Euler approximation of X in (4.1) by X^π in (4.4) is such that

$$\max_{n=0,\dots,N-1} \mathbb{E}\left[\sup_{t_n\le t\le t_{n+1}} \left|X_t - X_{t_n}^\pi\right|^2\right] \le \bar{C}_{x,\lambda}\,|\pi|, \tag{4.11}$$

where[2]

$$\bar{C}_{x,\lambda} = C(1+x^2)(1+\lambda)^4 e^{C(1+\lambda)^2} \tag{4.12}$$

and $C < \infty$ does not depend either on λ or on x.

[2]In all the sequel, $\bar{C}_{x,\lambda}^{2,e}$ (resp. $\bar{C}_{x,\lambda}$) will stand for any expression of the form (4.10) (resp. (4.12)), even for a different constant C.

Proof. This follows from classical arguments. For any $n < N$, define by extension $(X_t^\pi)_{0\le t\le T}$ as

$$X_t^\pi = X_{t_n}^\pi + \int_{t_n}^t b(X_{t_n}^\pi)ds + \int_{t_n}^t \sigma(X_{t_n}^\pi)dW_s + \int_{t_n}^t \gamma(X_{t_n}^\pi)dN_s^p, \quad \forall t \in [t_n, t_{n+1}).$$

Set $t_s^\pi := \max\{t_n, n = 0, \dots, N-1 : t_n \le s\}$, so that

$$X_t^\pi = x + \int_0^t b(X_{t_s^\pi}^\pi)ds + \int_0^t \sigma(X_{t_s^\pi}^\pi)dW_s + \int_0^t \gamma(X_{t_s^\pi}^\pi)dN_s^p, \quad \forall t \in [0, T].$$

From (H_X), Jensen's and Doob's inequalities and Gronwall's lemma, we easily prove in the same way as in Lemma 4.2.1 that

$$\mathbb{E}\left[\sup_{0\le t\le T} |X_t^\pi|^2\right] \le \bar{C}_{x,\lambda}^{2,e}. \tag{4.13}$$

Denote by $\delta b_s = b(X_s) - b(X_{t_s^\pi}^\pi)$ and $\delta\sigma_s$, $\delta\gamma_s$ in the same way. Using (H_X), Jensen's and Burkholder-Davis-Gundy's inequalities, we get $\forall t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t} |X_s - X_s^\pi|^2\right] \\
\le\ & 3\mathbb{E}\int_0^t |\delta b_s + \delta\gamma_s\lambda|^2 ds + 3\mathbb{E}\int_0^t |\delta\sigma_s|^2 ds + 3\mathbb{E}\int_0^t |\delta\gamma_s|^2 \lambda ds \\
\le\ & C\left(1+\lambda+\lambda^2\right)\int_0^t \mathbb{E}\left|X_s - X_{t_s^\pi}^\pi\right|^2 ds \\
\le\ & 2C\left(1+\lambda+\lambda^2\right)\Big(\max_{n<N}\mathbb{E}\left[\sup_{t_n\le s\le t_{n+1}} \left|X_s^\pi - X_{t_n}^\pi\right|^2\right] + \int_0^t \mathbb{E}\,|X_s - X_s^\pi|^2 ds\Big).
\end{aligned} \tag{4.14}$$

Now, using again (H_X), Jensen's and Burkholder-Davis-Gundy's inequalities and (4.13), we get $\forall n < N$,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t_n\le s\le t_{n+1}} \left|X_s^\pi - X_{t_n}^\pi\right|^2\right] &\le 3T\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |b(X_s^\pi) + \gamma(X_s^\pi)\lambda|^2 ds\right] + 3\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |\sigma(X_s^\pi)|^2 ds\right] \\
&\quad + 3\mathbb{E}\left[\int_{t_n}^{t_{n+1}} |\gamma(X_s^\pi)|^2 \lambda ds\right] \\
&\le C\left(1+\lambda+\lambda^2\right)|\pi| + C\int_{t_n}^{t_{n+1}} \left(1 + \mathbb{E}\,|X_s^\pi|^2\right) ds \\
&\le C(1+x^2)(1+\lambda)^2 e^{C(1+\lambda)^2}|\pi|.
\end{aligned} \tag{4.15}$$

Replacing (4.15) in (4.14) leads to

$$\mathbb{E}\left[\sup_{0\le s\le t} |X_s - X_s^\pi|^2\right] \le C(1+x^2)(1+\lambda)^4 e^{C(1+\lambda)^2}|\pi| + C\left(1+\lambda\right)^2 \int_0^t \mathbb{E}\left[\sup_{0\le r\le s} |X_r - X_r^\pi|^2\right] ds$$

and Gronwall's lemma implies thus

$$\mathbb{E}\left[\sup_{0\le s\le T} |X_s - X_s^\pi|^2\right] \le \bar{C}_{x,\lambda}\ |\pi|.$$

We get finally the expected result by again (4.15) and

$$\forall n < N, \quad \mathbb{E}\left[\sup_{t_n\le s\le t_{n+1}} \left|X_s - X_{t_n}^\pi\right|^2\right] \le 2\mathbb{E}\left[\sup_{t_n\le s\le t_{n+1}} |X_s - X_s^\pi|^2 + \sup_{t_n\le s\le t_{n+1}} \left|X_s^\pi - X_{t_n}^\pi\right|^2\right].$$

□

We present in Proposition 4.2.1 a first estimation of the errors $\mathcal{E}^\pi(Y^p)$, $\mathcal{E}^\pi(Z^p)$ and $\mathcal{E}^\pi(V^p)$. This result follows from the properties of the driver f^p of the penalized BSDE (4.2), see Lemma 4.2.3.

Lemma 4.2.3. Assume (H_Y). Then f^p defined in (4.3) is such that

(a) $f^p(0,0) \le C(1+\lambda p)$,

(b) it is Lipschitz in x with a Lipschitz coefficient proportional to $(1+\lambda p)$,

(c) it is Lipschitz in v with a Lipschitz coefficient proportional to λK_p, with[3] $K_p := p \vee 1$,

(d) it satisfies the following growth condition:

$$\forall (x,v) \in \mathbb{R}^d \times \mathbb{R}, \quad |f^p(x,v)| \le C\left[(1+\lambda p) + (1+\lambda p)\,|x| + \lambda K_p\,|v|\right].$$

Proof. Straightforward from 1-Lipschitz property of $y \mapsto (y)^+$. □

Proposition 4.2.1. *Assume* (H). *Set for any $n < N$ and $t \in [t_n, t_{n+1})$*

$$\bar{Z}^p_t := \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} Z^p_s ds\right], \qquad \bar{V}^p_t := \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} V^p_s ds\right]. \tag{4.16}$$

If $|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right)$, then

$$\begin{aligned}
|\mathcal{E}^\pi(Y^p)|^2 &\le D^1_x(\lambda,p)|\pi| + 2\max_{n<N} \mathbb{E}\left[\sup_{t_n \le t \le t_{n+1}} \left|Y^p_t - Y^p_{t_n}\right|^2\right] + D^2(\lambda,p)\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}, \\
|\mathcal{E}^\pi(V^p)|^2 &\le D^3_x(\lambda,p)|\pi| + D^4(\lambda,p)\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}, \\
|\mathcal{E}^\pi(Z^p)|^2 &\le D^3_x(\lambda,p)|\pi| + 2\left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)} + D^4(\lambda,p)\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)},
\end{aligned}$$

where

$$\begin{aligned}
D^1_x(\lambda,p) &= \bar{C}_{x,\lambda}\left(1 + \tfrac{(1+\lambda p)^2}{\lambda p^2}\right) D^2(\lambda,p) \\
D^3_x(\lambda,p) &= \bar{C}_{x,\lambda}\left(1 + \tfrac{(1+\lambda p)^2}{\lambda p^2}\right) D^4(\lambda,p)
\end{aligned}$$

and $\bar{C}_{x,\lambda}$ has already been defined (4.12). *In addition,*

$$D^2(\lambda,p) = \mathcal{O}\left(e^{C\lambda p^2}\right), \qquad D^4(\lambda,p) = \mathcal{O}\left(\lambda p^2 e^{C\lambda p^2}\right) \quad \text{as } p \to +\infty,$$

so that

$$D^1_x(\lambda,p) = \mathcal{O}\left(\bar{C}_{x,\lambda}(1+\lambda)e^{C\lambda p^2}\right), \qquad D^3_x(\lambda,p) = \mathcal{O}\left(\bar{C}_{x,\lambda}(1+\lambda)\lambda p^2 e^{C\lambda p^2}\right) \quad \text{as } p \to +\infty.$$

[3] From now and without loss of generality, we will assume that $p > 1$, namely $K_p = p$. This is consistent with Remark (4.1.3).

Proof. By the representation theorem, there exists two processes $(Z^{p,\pi}, V^{p,\pi}) \in \mathrm{L}^2(W) \times \mathrm{L}^2(N)$ such that $\forall n = 0, \dots, N-1$,

$$\bar{Y}^{p,\pi}_{t_{n+1}} - \mathbb{E}_{t_n}\left[\bar{Y}^{p,\pi}_{t_{n+1}}\right] = \int_{t_n}^{t_{n+1}} Z^{p,\pi}_s dW_s + \int_{t_n}^{t_{n+1}} V^{p,\pi}_s d\tilde{N}_s$$

so that $(\bar{Z}^{p,\pi}, \bar{V}^{p,\pi})$ defined in (4.5) satisfy, $\forall t \in [t_n, t_{n+1})$,

$$\bar{Z}^{p,\pi}_t = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} Z^{p,\pi}_s ds\right], \qquad \bar{V}^{p,\pi}_{t_n} = \frac{1}{\Delta t_{n+1}} \mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} V^{p,\pi}_s ds\right]. \tag{4.17}$$

Besides, define $Y^{p,\pi} \in \mathcal{S}^2$ such that $\forall n = 0, \dots, N-1$, $\forall t \in [t_n, t_{n+1})$,

$$Y^{p,\pi}_t = \bar{Y}^{p,\pi}_{t_n} - f^p(X^\pi_{t_n}, \bar{V}^{p,\pi}_{t_n})(t - t_n) + \int_{t_n}^t Z^{p,\pi}_s dW_s + \int_{t_n}^t V^{p,\pi}_s d\tilde{N}_s.$$

It follows from this definition that $Y^{p,\pi}_{t_n} = \bar{Y}^{p,\pi}_{t_n}$ and together with (4.2), we get $\forall t \in [t_n, t_{n+1})$:

$$\begin{aligned} Y^p_t &= Y^p_{t_{n+1}} + \int_t^{t_{n+1}} f^p(X_{s^-}, V^p_s) ds - \int_t^{t_{n+1}} Z^p_s dW_s - \int_t^{t_{n+1}} V^p_s d\tilde{N}_s, \\ Y^{p,\pi}_t &= Y^{p,\pi}_{t_{n+1}} + f^p(X^\pi_{t_n}, \bar{V}^{p,\pi}_{t_n})(t_{n+1} - t) - \int_t^{t_{n+1}} Z^{p,\pi}_s dW_s - \int_t^{t_{n+1}} V^{p,\pi}_s d\tilde{N}_s. \end{aligned}$$

Set $\delta A^p = A^p - A^{p,\pi}$ for $A = Y, Z, V$ and use shorthand notation

$$\forall t \in [t_n, t_{n+1}), \quad \delta f^p_t = f^p(X_t, V^p_t) - f^p(X^\pi_{t_n}, \bar{V}^{p,\pi}_{t_n})$$

so that

$$\delta Y^p_t = \delta Y^p_{t_{n+1}} + \int_t^{t_{n+1}} \delta f^p_s ds - \int_t^{t_{n+1}} \delta Z^p_s dW_s - \int_t^{t_{n+1}} \delta V^p_s d\tilde{N}_s.$$

Applying Itô lemma to $|\delta Y^p_t|^2$ on $[t, t_{n+1})$, remarking that $\Delta \delta Y^p_t = \delta V^p_t$ and taking the expectation, we get

$$\mathbb{E}|\delta Y^p_t|^2 + \mathbb{E}\int_t^{t_{n+1}} |\delta Z^p_s|^2 ds + \mathbb{E}\int_t^{t_{n+1}} |\delta V^p_s|^2 \lambda ds = \mathbb{E}|\delta Y^p_{t_{n+1}}|^2 + \mathbb{E}\int_t^{t_{n+1}} 2\delta Y^p_s \delta f^p_s ds. \tag{4.18}$$

With properties of f^p in Lemma 4.2.3, we have

$$|\delta f^p_s| \le C(1 + \lambda p)|X_s - X^\pi_{t_n}| + \lambda p |V^p_s - \bar{V}^{p,\pi}_{t_n}|.$$

Thus, using inequality $\forall \alpha > 0, 2ab \le \alpha a^2 + \frac{1}{\alpha} b^2$ and estimate (4.11), it follows from (4.18) that

$$\begin{aligned} & \mathbb{E}|\delta Y^p_t|^2 + \mathbb{E}\int_t^{t_{n+1}} |\delta Z^p_s|^2 ds + \mathbb{E}\int_t^{t_{n+1}} |\delta V^p_s|^2 \lambda ds \\ \le\ & \mathbb{E}|\delta Y^p_{t_{n+1}}|^2 + \alpha \int_t^{t_{n+1}} \mathbb{E}|\delta Y^p_s|^2 ds + \tfrac{1}{\alpha}\int_t^{t_{n+1}} \mathbb{E}|\delta f^p_s|^2 ds \\ \le\ & \mathbb{E}|\delta Y^p_{t_{n+1}}|^2 + \alpha \int_t^{t_{n+1}} \mathbb{E}|\delta Y^p_s|^2 ds + \tfrac{\bar{C}_{x,\lambda}}{\alpha}(1 + \lambda p)^2 |\pi|^2 + \tfrac{\lambda p^2}{\alpha}\int_t^{t_{n+1}} \mathbb{E}|V^p_s - \bar{V}^{p,\pi}_{t_n}|^2 \lambda ds. \end{aligned}$$

With definitions (4.16) and (4.17) and Jensen's inequality, we have

$$\begin{aligned} \mathbb{E}|V^p_s - \bar{V}^{p,\pi}_{t_n}|^2 &\le 2\left(\mathbb{E}|V^p_s - \bar{V}^p_{t_n}|^2 + \mathbb{E}|\bar{V}^p_{t_n} - \bar{V}^{p,\pi}_{t_n}|^2\right) \\ &= 2\left(\mathbb{E}|V^p_s - \bar{V}^p_{t_n}|^2 + \mathbb{E}|\frac{1}{\Delta t_{n+1}}\mathbb{E}_{t_n}\int_{t_n}^{t_{n+1}} \delta V^p_t dt|^2\right) \\ &\le 2\left(\mathbb{E}|V^p_s - \bar{V}^p_{t_n}|^2 + \frac{1}{\Delta t_{n+1}}\int_{t_n}^{t_{n+1}} \mathbb{E}|\delta V^p_t|^2 dt\right) \end{aligned}$$

so that

$$\int_t^{t_{n+1}} \mathbb{E}|V_s^p - \bar{V}_{t_n}^{p,\pi}|^2 \lambda ds \leq 2 \left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} + 2 \left\|\delta V^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)}.$$

In consequence, we obtain

$$\mathbb{E}|\delta Y_t^p|^2 \leq \mathbb{E}|\delta Y_t^p|^2 + \left\|\delta Z^p\right\|^2_{\mathrm{L}^2_{[t,t_{n+1}]}(W)} + \left\|\delta V^p\right\|^2_{\mathrm{L}^2_{[t,t_{n+1}]}(N)} \leq B_{t_n} + \alpha \int_t^{t_{n+1}} \mathbb{E}|\delta Y_s^p|^2 ds, \quad (4.19)$$

where

$$B_{t_n} = \mathbb{E}|\delta Y_{t_{n+1}}^p|^2 + \tfrac{\bar{C}_{x,\lambda}}{\alpha}(1+\lambda p)^2|\pi|^2 + \tfrac{2\lambda p^2}{\alpha}\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} + \tfrac{2\lambda p^2}{\alpha}\left\|\delta V^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)}.$$

Applying Gronwall's lemma to $t \to \mathbb{E}|\delta Y_t^p|^2$ in (4.19), we get

$$\mathbb{E}|\delta Y_t^p|^2 \leq B_{t_n} e^{\alpha|\pi|}.$$

Taking $\alpha = 8\lambda p^2$ and $|\pi| < \frac{1}{16\lambda p^2}$, we have $\frac{2\lambda p^2}{\alpha}(1+2\alpha|\pi|) < \frac{1}{2}$ and $e^{\alpha|\pi|} < e^{\frac{1}{2}} < 2$. Thus, replacing previous estimation in equation (4.19) and taking $t = t_n$,

$$\mathbb{E}|\delta Y_{t_n}^p|^2 + \left\|\delta Z^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)} + \left\|\delta V^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} \leq (1+\alpha|\pi|e^{\alpha|\pi|})B_{t_n} \leq (1+2\alpha|\pi|)B_{t_n}.$$

Hence,

$$\begin{aligned}\mathbb{E}|\delta Y_{t_n}^p|^2 &\leq \mathbb{E}|\delta Y_{t_n}^p|^2 + \left\|\delta Z^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)} + \frac{1}{2}\left\|\delta V^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} \\ &\leq (1+16\lambda p^2|\pi|)\mathbb{E}|\delta Y_{t_{n+1}}^p|^2 + \frac{1}{4}\bar{C}_{x,\lambda}\frac{(1+\lambda p)^2}{\lambda p^2}|\pi|^2 + \frac{1}{2}\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)}. \quad (4.20)\end{aligned}$$

Iterating on t_n until $t_N = T$ (recall that $|\pi| = \mathcal{O}(\frac{1}{N})$), using the linear growth of g and estimate (4.11), we get

$$\mathbb{E}|\delta Y_{t_n}^p|^2 \leq \bar{C}_{x,\lambda}(1+16\lambda p^2|\pi|)^{\frac{1}{|\pi|}}\left(1+\tfrac{(1+\lambda p)^2}{\lambda p^2}\right)|\pi| + \tfrac{1}{2}(1+16\lambda p^2|\pi|)^{\frac{1}{|\pi|}}\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}.$$

But

$$(1+16\lambda p^2|\pi|)^{\frac{1}{|\pi|}} \leq \exp\left\{\tfrac{1}{|\pi|}16\lambda p^2|\pi|\right\} \leq Ce^{C\lambda p^2}$$

implies

$$\mathbb{E}|\delta Y_{t_n}^p|^2 \leq \bar{C}_{x,\lambda}\left(1+\tfrac{(1+\lambda p)^2}{\lambda p^2}\right)e^{C\lambda p^2}|\pi| + Ce^{C\lambda p^2}\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)},$$

which, together with

$$|\mathcal{E}^\pi(Y^p)|^2 \leq 2\max_{n<N}\mathbb{E}\left[\sup_{t_n\leq t\leq t_{n+1}}\left|Y_t^p - Y_{t_n}^p\right|^2\right] + 2\max_{n<N}\mathbb{E}|\delta Y_{t_n}^p|^2,$$

allows us to conclude for $\mathcal{E}^\pi(Y^p)$. Pluging previous estimation in (4.20), we get

$$\begin{aligned}&\mathbb{E}|\delta Y_{t_n}^p|^2 + \left\|\delta Z^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)} + \tfrac{1}{2}\left\|\delta V^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} \\ \leq\ &\mathbb{E}|\delta Y_{t_{n+1}}^p|^2 + 16\lambda p^2|\pi|\mathbb{E}|\delta Y_{t_{n+1}}^p|^2 + \tfrac{1}{4}\bar{C}_{x,\lambda}\tfrac{(1+\lambda p)^2}{\lambda p^2}|\pi|^2 + \tfrac{1}{2}\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} \\ \leq\ &\mathbb{E}|\delta Y_{t_{n+1}}^p|^2 + \bar{C}_{x,\lambda}\left(1+\tfrac{(1+\lambda p)^2}{\lambda p^2}\right)\left(1+\lambda p^2 e^{C\lambda p^2}\right)|\pi|^2 + \tfrac{1}{2}\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} \\ &+C\lambda p^2 e^{C\lambda p^2}|\pi|\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}.\end{aligned}$$

Now, summing up over $n = 0, \dots, N-1$ in last equation, we get

$$\begin{aligned} & \|\delta Z^p\|^2_{\mathrm{L}^2(W)} + \tfrac{1}{2}\|\delta V^p\|^2_{\mathrm{L}^2(N)} \\ \leq\ & \bar{C}_{x,\lambda}\left(1 + \tfrac{(1+\lambda p)^2}{\lambda p^2}\right)\left(1 + \lambda p^2 e^{C\lambda p^2}\right)|\pi| + C\left(1 + \lambda p^2 e^{C\lambda p^2}\right)\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}, \end{aligned}$$

which, together with

$$\begin{aligned} |\mathcal{E}^\pi(Z^p)|^2 &\leq 2\left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)} + 2\|\delta Z^p\|^2_{\mathrm{L}^2(W)} \\ |\mathcal{E}^\pi(V^p)|^2 &\leq 2\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)} + 2\|\delta V^p\|^2_{\mathrm{L}^2(N)}, \end{aligned}$$

lead to the expected estimates for $\mathcal{E}^\pi(Z^p)$ and $\mathcal{E}^\pi(V^p)$. □

The estimations for errors $\mathcal{E}^\pi(Y^p)$, $\mathcal{E}^\pi(Z^p)$ and $\mathcal{E}^\pi(V^p)$ in Proposition 4.2.1 involves three residual errors linked to the path-regularity of the continuous time solution (Y^p, Z^p, V^p), namely

$$\max_{n=0,\dots,N-1} \mathbb{E}\left[\sup_{t_n \leq t \leq t_{n+1}} \left|Y^p_t - Y^p_{t_n}\right|^2\right], \quad \left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}, \quad \left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)}, \tag{4.21}$$

in which $\bar{Z}^p$ and $\bar{V}^p$ have been defined in (4.16). We provide in next Section 4.3 upper bounds of order $|\pi|$ on the two first errors and of order $|\pi|^{\frac{1}{2}}$ on the last error.

4.3 Estimates involving the path-regularity of the continuous-time solution

We use standard regularization arguments to smoothen the coefficients of FBSDE (4.1)-(4.2). Consider a $\mathcal{C}^\infty_b$ density q with compact support on $\mathbb{R}^d$. For any FBSDE coefficient $\xi \in \{b, \sigma, \gamma, g\}$ and $k > 0$, we introduce its regularized version defined by[4]

$$\xi^k(x) = k^d \int_{\mathbb{R}^d} \xi(\bar{x}) q(k[\bar{x} - x]) d\bar{x}. \tag{4.22}$$

In a same way, consider a $\mathcal{C}^\infty_b(\mathbb{R}^{d+1})$ density d with compact support on $\mathbb{R}^{d+1}$ and introduce the regularized version $f^{p,k}$ of the BSDE driver f^p defined in (4.3), namely

$$f^{p,k}(x, v) = k^{d+1} \int_{\mathbb{R}^{d+1}} f^p(\bar{x}, \bar{v}) d(k[\bar{x} - x], k(\bar{v} - v)) d\bar{x} d\bar{v}. \tag{4.23}$$

Remark 4.3.1. One can see $\xi^k(x)$ as the convolution of ξ and $y \to kq(ky)$ at point x. In dimension $d = 1$, a typical choice for q would be the following:

$$kq(ky) = \begin{cases} ke^{-1/(1-|ky|^2)} & \text{if } |y| < \frac{1}{k} \\ 0 & \text{if } |y| \geq \frac{1}{k} \end{cases} \longrightarrow \delta_0, \quad \text{as } k \to +\infty.$$

Dividing this map by its integral over $\mathbb{R}$, this constitutes a *positive and symmetric mollifier*.

[4] For a $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $k[x] := (kx_1, kx_2, \dots, kx_d)$.

A regularization parameter $k > 0$ being fixed, we shall denote by $(Y^{p,k}, Z^{p,k}, V^{p,k})$ the unique solution to BSDE with jumps

$$Y_t^{p,k} = g^k(X_T^k) + \int_t^T f^{p,k}(X_s^k, V_s^{p,k})ds - \int_t^T Z_s^{p,k}dW_s - \int_t^T V_s^{p,k}d\tilde{N}_s, \forall 0 \le t \le T, \quad (4.24)$$

in which X^k satisfies SDE

$$X_t^k = x + \int_0^t b^k(X_s^k)ds + \int_0^t \sigma^k(X_s^k)dW_s + \int_0^t \gamma^k(X_s^k)dN_s, \forall 0 \le t \le T. \quad (4.25)$$

$(X^k, Y^{p,k}, Z^{p,k}, V^{p,k})$ constitues the counterpart of solution (X, Y^p, Z^p, V^p) to (4.1)-(4.2) when the FBSDE coefficients are replaced by their regularized versions.

The regularized FBSDE coefficients $b^k, \sigma^k, \gamma^k, g^k$ and $f^{p,k}$ have the benefit to keep the same regularity properties than the original coefficients, but to be also smoother than them: under (H), they still verify conditions in (H) but belong to $\mathcal{C}_b^1$ and under (H′), they still verify conditions in (H′) but belong to $\mathcal{C}_b^2$. Lemma 4.3.1 resume more precisely these properties.

This will further allow us to use Malliavin differentiation arguments which provide representations to processes $Z^{p,k}$ and $V^{p,k}$ and make easy the estimation of the same errors as in (4.21) when replacing $(Y^p, Z^p, V^p, \bar{Z}^p, \bar{V}^p)$ by its approximate regularized version. That is $(Y^{p,k}, Z^{p,k}, V^{p,k}, \bar{Z}^{p,k}, \bar{V}^{p,k})$, in which $\bar{Z}^{p,k}$ and $\bar{V}^{p,k}$ are defined similarly to (4.26), namely for any $n < N$ and $t \in [t_n, t_{n+1})$,

$$\bar{Z}_t^{p,k} := \frac{1}{\Delta t_{n+1}}\mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} Z_s^{p,k}ds\right], \qquad \bar{V}_t^{p,k} := \frac{1}{\Delta t_{n+1}}\mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} V_s^{p,k}ds\right]. \quad (4.26)$$

Besides, it is straightforward from their definitions that the regularized coefficients $b^k, \sigma^k, \gamma^k, g^k$ and $f^{p,k}$ converges pointwise to b, σ, γ, g and f^p respectively as k goes to infinity. This intuitively implies that the solution $(X^k, Y^{p,k}, Z^{p,k}, V^{p,k})$ to (4.25)-(4.24) converges to the solution (X, Y^p, Z^p, V^p) to (4.1)-(4.2) in $\mathcal{S}^2 \times \mathcal{S}^2 \times \mathrm{L}^2(W) \times \mathrm{L}^2(N)$ as $k \to +\infty$, see Proposition 4.3.1. This last argument will allow us to complete our estimation of regularity errors in (4.21).

Lemma 4.3.1. Let $\xi \in \{b, \sigma, \gamma, g\}$, f^p be defined in (4.3), $k > 0$, ξ^k be defined in (4.22) and $f^{p,k}$ in (4.23). Let assumption (H) be satisfied. Then, there exist a constant C which does not depend either on λ, p or k such that

(i) We have
$$\left\|\xi - \xi^k\right\|_\infty \le \frac{C}{k}, \quad \left\|f^p - f^{p,k}\right\|_\infty \le ((1 + \lambda p) + \lambda K_p)\frac{C}{k},$$
recalling $K_p := p \vee 1$.

(ii) ξ^k and $f^{p,k}$ are bounded at 0 as soon as $k \ge 1$, namely
$$\forall k \ge 1, \quad \left|\xi^k(0)\right| \le C, \quad \left|f^{p,k}(0,0)\right| \le C(1 + \lambda p).$$

(iii) ξ^k is Lipschitz continuous and $f^{p,k}$ is Lipschitz in x with a Lipschitz coefficient proportional to $(1 + \lambda p)$ and Lipchitz in v with a Lipschitz coefficient proportional to λK_p.

(iv) ξ^k and $f^{p,k}$ are in $\mathcal{C}^1_b$, namely[5]

$$\left\|\nabla \xi^k\right\|_\infty \leq C, \quad \left\|\nabla_x f^{p,k}\right\|_\infty \leq C(1+\lambda p), \quad \left\|\nabla_v f^{p,k}(x,v)\right\|_\infty \leq C\lambda K_p.$$

(v) Let stronger assumption (H′) be satisfied. Then ξ^k also verifies the conditions in (H′) and belongs to $\mathcal{C}^2_b$, and this holds uniformly in k. Besides, $f^{p,k}$ also verifies the conditions in (H′) and belongs to $\mathcal{C}^2_b$ but with coefficients depending on k, namely

- $\nabla_x f^{p,k}$ and $\nabla_v f^{p,k}$ are Lipshitz in x with a Lipschitz coefficient prop. to $(1+\lambda p)k$,
- $\nabla_x f^{p,k}$ and $\nabla_v f^{p,k}$ are Lipshitz in v with a Lipschitz coefficient prop. to $\lambda K_p k$,
- $\left\|\nabla^2_{xx} f^p\right\|_\infty + \left\|\nabla^2_{xv} f^p\right\|_\infty \leq C(1+\lambda p)k$,
- $\left\|\nabla^2_{vv} f^p\right\|_\infty + \left\|\nabla^2_{vx} f^p\right\|_\infty \leq C\lambda K_p k$.

Proof. This comes from the properties of f^p in Lemma 4.2.3 (which also hold for $f^{p,k}$, see point (iii)) and straightforward computations. Let us show for example the properties involving $\nabla_x f^p$ in (iv) and (v). Since $f^{p,k}$ is $C(1+\lambda p)$-Lipschitz in x, $\nabla_x f^{p,k}$ is bounded by $C(1+\lambda p)$ by characterization of Lipschitz and differentiable maps. In addition,

$$\nabla_x f^{p,k}(x,v) = -k^{d+2} \int_{\mathbb{R}^{d+1}} f^p(\bar{x},\bar{v}) \nabla_x d(k[\bar{x}-x], k(\bar{v}-v)) d\bar{x} d\bar{v},$$

which implies, by $C(1+\lambda p)$-Lipschitz property of f^p in x, $\forall (x,x',v) \in \mathbb{R}^{2d+1}$,

$$\begin{aligned} & \left|\nabla_x f^{p,k}(x,v) - \nabla_x f^{p,k}(x',v)\right| \\ = \; & \frac{k^{d+2}}{k^{d+1}} \left| \int_{\mathbb{R}^{d+1}} f^p(x+\frac{1}{k}[y], v+\frac{u}{k}) \nabla_x d(y,u) dy du - \int_{\mathbb{R}^{d+1}} f^p(x'+\frac{1}{k}[y], v+\frac{u}{k}) \nabla_x d(y,u) dy du \right| \\ \leq \; & C(1+\lambda p)k \left|x-x'\right|, \end{aligned}$$

leading to

$$\left|\nabla^2_{xx} f^p(x,v)\right| \leq C(1+\lambda p)k.$$

In a same way, by $C\lambda K_p$-Lipschitz property of f^p in v, we get, $\forall (x,v,v') \in \mathbb{R}^{d+2}$,

$$\begin{aligned} & \left|\nabla_x f^{p,k}(x,v) - \nabla_x f^{p,k}(x,v')\right| \\ = \; & k \left| \int_{\mathbb{R}^{d+1}} f^p(x+\frac{1}{k}[y], v+\frac{u}{k}) \nabla_x d(y,u) dy du - \int_{\mathbb{R}^{d+1}} f^p(x+\frac{1}{k}[y], v'+\frac{u}{k}) \nabla_x d(y,u) dy du \right| \\ \leq \; & C\lambda K_p k \left|v-v'\right|, \end{aligned}$$

leading to

$$\left|\nabla^2_{vx} f^p(x,v)\right| \leq C\lambda K_p k.$$

□

Proposition 4.3.1 provides an estimation of the regularization error.

[5]When the lower index of the gradient ∇ is not specified and its argument is a function of x only, we may write $\nabla := \nabla_x$.

Proposition 4.3.1. *Let assumption* (H) *be satisfied and* $k \geq 1$. *Then*

$$\left\|X - X^k\right\|^2_{\mathcal{S}^2} \leq \bar{C}^{2,e}_\lambda \frac{1}{k^2}$$

$$\left\|Y^p - Y^{p,k}\right\|^2_{\mathcal{S}^2} + \left\|Z^p - Z^{p,k}\right\|^2_{\mathrm{L}^2(W)} + \left\|V^p - V^{p,k}\right\|^2_{\mathrm{L}^2(N)} \leq R(\lambda,p)\frac{1}{k^2},$$

where[6]

$$\bar{C}^{2,e}_\lambda = C(1+\lambda)^2 e^{C(1+\lambda)^2} \tag{4.27}$$

and

$$R(\lambda,p) := \bar{C}^{2,e}_\lambda e^{C\lambda p^2}\left(\lambda p^2 + (1+\frac{1}{\lambda})\left((1+\lambda p)^2 + \lambda p^2 (\lambda p)^2\right)\right). \tag{4.28}$$

Proof. This comes from points (i) of Lemma 4.3.1. See the proof of Lemma 4.A.2 for a more detailed reasoning using the same kind of arguments. Denote for ease of notation

$$\begin{array}{rl} & \delta X = X - X^k, \\ & \delta f^p = f^p - f^{p,k}, \\ \forall A \in \{Y,Z,V\}, & \delta A^p = A^p - A^{p,k}. \end{array}$$

Divide $[0,T]$ in $K \geq 1$ intervals $([s_l, s_{l+1}])_{l=0,\ldots,K-1}$ with same length $\delta_K = \frac{T}{K}$. BSDEs (4.2) and (4.24) give, for any $s_l \leq s \leq s_{l+1}$,

$$\delta Y^p_s + \int_s^{s_{l+1}} \delta Z^p_r dW_r + \int_s^{s_{l+1}} \delta V^p_r d\tilde{N}_r = \delta Y^p_{s_{l+1}} + \int_s^{s_{l+1}} \left(f^p(X_r, V^p_r) - f^{p,k}(X^k_r, V^{p,k}_r)\right) dr,$$

so that

$$\begin{array}{rl} & \|\delta Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \|\delta Z^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \|\delta V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq & C\mathbb{E}\left[\left|\delta Y^p_{s_{l+1}}\right|^2 + \left|\int_{s_l}^{s_{l+1}} \left|f^p(X_r,V^p_r) - f^{p,k}(X^k_r,V^{p,k}_r)\right| dr\right|^2\right]. \end{array}$$

But Lemma 4.3.1-(i) and Lipschitz property of f^p in Lemma 4.2.3 imply

$$\begin{aligned} \left|f^p(X_r,V^p_r) - f^{p,k}(X^k_r,V^{p,k}_r)\right| &\leq \left|f^p(X_r,V^p_r) - f^p(X^k_r,V^{p,k}_r)\right| + \left|\delta f^p(X^k_r,V^{p,k}_r)\right| \\ &\leq C(1+\lambda p)\left|\delta X_r\right| + C\lambda p\left|\delta V^p_r\right| + C\left[(1+\lambda p) + \lambda p\right]\frac{1}{k}. \end{aligned}$$

By Cauchy-Schwartz's inequality, we get thus

$$\begin{array}{rl} & \|\delta Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \|\delta Z^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \|\delta V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq & C\mathbb{E}\left|Y^p_{s_{l+1}}\right|^2 + C\delta_K^2(1+\lambda p)^2\|\delta X\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + C\delta_K\lambda p^2\|\delta V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ & + C\delta_K^2\left[(1+\lambda p)^2 + (\lambda p)^2\right]\frac{1}{k^2} \end{array}$$

[6] In all the sequel, $\bar{C}^{2,e}_\lambda$ will stand for any expression of this form, even for a different constant C.

Choosing $\delta_K = \frac{1}{2C\lambda p^2}$, this leads to

$$\begin{aligned} & \|\delta Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \|\delta Z^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\|\delta V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq\ & C\mathbb{E}\left|Y^p_{s_{l+1}}\right|^2 + C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\|\delta X\|^2_{\mathcal{S}^2} + C\left(\frac{(1+\lambda p)^2}{(\lambda p^2)^2}+\lambda\right)\frac{1}{k^2}. \end{aligned} \tag{4.29}$$

Iterating on s_l until $s_K = T$ in equation (4.29) (recall $K = 2CT\lambda p^2$), we get, for any $l = 0,\dots,K-1$,

$$\begin{aligned} \|\delta Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} &\leq C^{K-l}\mathbb{E}\left|g(X_T) - g^k(X^k_T)\right|^2 + (K-l)C^{K-l}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\|\delta X\|^2_{\mathcal{S}^2} \\ &\quad + (K-l)C^{K-l}\left(\frac{(1+\lambda p)^2}{(\lambda p^2)^2}+\lambda\right)\frac{1}{k^2} \\ &\leq C^K\left(\frac{1}{k^2} + \mathbb{E}\left|\delta X_T\right|^2\right) + C^K\frac{(1+\lambda p)^2}{\lambda p^2}\|\delta X\|^2_{\mathcal{S}^2} + C^K\left(\frac{(1+\lambda p)^2}{\lambda p^2} + (\lambda p)^2\right)\frac{1}{k^2} \\ &\leq C^K\left(1 + \frac{(1+\lambda p)^2}{\lambda p^2}\right)\|\delta X\|^2_{\mathcal{S}^2} + C^K\left(1 + \frac{(1+\lambda p)^2}{\lambda p^2} + (\lambda p)^2\right)\frac{1}{k^2}, \end{aligned} \tag{4.30}$$

which leads to

$$\begin{aligned} \|\delta Y^p\|^2_{\mathcal{S}^2} &\leq K \max_{l=0,\dots,K-1}\|\delta Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \\ &\leq e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)\|\delta X\|^2_{\mathcal{S}^2} + e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2 + \lambda p^2(\lambda p)^2\right)\frac{1}{k^2}. \end{aligned}$$

Summing up over $l = 0,\dots,K-1$ in equation (4.29) leads, together with (4.30), to

$$\begin{aligned} & \|\delta Z^p\|^2_{\mathrm{L}^2(W)} + \frac{1}{2}\|\delta V^p\|^2_{\mathrm{L}^2(N)} \\ \leq\ & C\sum_{l=0}^{K-1}\mathbb{E}\left|\delta Y^p_{s_{l+1}}\right|^2 + C\frac{(1+\lambda p)^2}{\lambda p^2}\|\delta X\|^2_{\mathcal{S}^2} + C\left(\frac{(1+\lambda p)^2}{\lambda p^2} + (\lambda p)^2\right)\frac{1}{k^2} \\ \leq\ & C^K\left(\lambda p^2 + (1+\frac{1}{\lambda})(1+\lambda p)^2\right)\|\delta X\|^2_{\mathcal{S}^2} \\ & + C^K\left(\lambda p^2 + (1+\frac{1}{\lambda})\left((1+\lambda p)^2 + \lambda p^2(\lambda p)^2\right)\right)\frac{1}{k^2}. \end{aligned}$$

It remains us to provide a bound on $\|\delta X\|^2_{\mathcal{S}^2}$ of order $\frac{1}{k^2}$, which is easy with classical arguments. Set

$$\forall \xi \in \{b,\sigma,\gamma\}, \quad \delta\xi_s = \xi(X_s) - \xi^k(X^k_s).$$

SDEs (4.25) and (4.1) give, for any $t \leq s \leq T$,

$$\delta X_s = \int_0^s \left[\delta b_r + \delta\gamma_r\lambda\right]dr + \int_0^s \delta\sigma_r dW_r + \int_0^s \delta\gamma_r d\tilde{N}_r$$

By Lemma 4.3.1-(i), Doob's inequality and (H_X), we get then

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}|\delta X_s|^2\right] &\le 3T\mathbb{E}\int_0^t|\delta b_r+\delta\gamma_r\lambda|^2\,dr+3\mathbb{E}\int_0^t|\delta\sigma_r|^2\,dr+3\mathbb{E}\int_0^t|\delta\gamma_r|^2\,\lambda dr\\ &\le C\left(1+\lambda+\lambda^2\right)\mathbb{E}\int_0^t\left(\frac{1}{k^2}+|\delta X_s|^2\right)ds\\ &\le C\left(1+\lambda\right)^2\frac{1}{k^2}+C\left(1+\lambda\right)^2\int_0^t\mathbb{E}\left[\sup_{0\le r\le s}|\delta X_r|^2\right]ds,\end{aligned}$$

so that Gronwall's lemma implies

$$\|\delta X\|^2_{\mathcal{S}^2}\le C(1+\lambda)^2e^{C(1+\lambda)^2}\frac{1}{k^2}=\bar{C}^{2,e}_{\lambda}\frac{1}{k^2}$$

which leads to the expected bounds for $\|\delta Y^p\|^2_{\mathcal{S}^2}$, $\|\delta Z^p\|^2_{\mathrm{L}^2(W)}$ and $\|\delta V^p\|^2_{\mathrm{L}^2(N)}$. □

Proposition 4.3.2. *Let assumption* (H) *be satisfied and* $|\pi|=\mathcal{O}\left(\frac{1}{\lambda p^2}\right)$. *Then for any* $k\ge 1$,

$$\max_{n=0,\dots,N-1}\mathbb{E}\left[\sup_{t_n\le t\le t_{n+1}}\left|Y^{p,k}_t-Y^{p,k}_{t_n}\right|^2\right]\le E^1_x(\lambda,p)|\pi|.$$

Besides, there exists a version of $V^{p,k}$ *such that*

$$\max_{n=0,\dots,N-1}\sup_{t_n\le t\le t_{n+1}}\mathbb{E}\left|V^{p,k}_t-V^{p,k}_{t_n}\right|^2\le E^1_x(\lambda,p)|\pi|.$$

Assume in addition that (H′) *holds. Then, there exists a version of* $Z^{p,k}$ *such that*

$$\max_{n=0,\dots,N-1}\sup_{t_n\le t\le t_{n+1}}\mathbb{E}\left|Z^{p,k}_t-Z^{p,k}_{t_n}\right|^2\le\left(E^2_x(\lambda,p)+E^3_x(\lambda,p)k^2\right)|\pi|.$$

where $E^i_x(\lambda,p), i\in\{1,2,3\}$ *do not depend either on* k *or* $|\pi|$*:*

$$E^1_x(\lambda,p)=\bar{C}^{2,e}_{x,\lambda}(1+\lambda)e^{C\lambda p^2}\left((1+\lambda p)^2+\lambda p^2\right) \tag{4.31}$$

$$E^2_x(\lambda,p)=\left(\bar{C}^{2,e}_{x,\lambda}+\bar{C}^{4,e}_{x,\lambda}\right)(1+\lambda)e^{C\lambda p^2}\left((1+\lambda p)^2+\lambda p^2\right) \tag{4.32}$$

$$E^3_x(\lambda,p)=\bar{C}^{4,e}_{x,\lambda}\,e^{C\lambda p^2}\left(1+\frac{1}{\lambda}+\lambda^3p^4\right)\left((1+\lambda p)^2+(\lambda p)^2\right) \tag{4.33}$$

and[7]

$$\bar{C}^{q,e}_{x,\lambda}=C(1+x^q)(1+\lambda)^qe^{C(1+\lambda)^q}. \tag{4.34}$$

Proof. This proof is deferred to Appendix 4.A.2. □

As a consequence of Proposition 4.3.2, we get the following bound on regularity errors of approximate regularized solution $(Y^{p,k},Z^{p,k},V^{p,k})$.

[7] As usual, the expressions for $\bar{C}^{q,e}_{x,\lambda}, q\in\{2,4\}$ hold even for a different constant C. Notice that the definition in (4.34) is consistent with the expression (4.10).

Corollary 4.3.1. *Let assumption* (H) *be satisfied and* $|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right)$ *and* $k \geq 1$. *Then, there exists a version of* $V^{p,k}$ *such that*

$$\max_{n=1,\dots,N-1} \mathbb{E}\left[\sup_{t_n \leq t \leq t_{n+1}} \left|Y_t^{p,k} - Y_{t_n}^{p,k}\right|^2\right] \leq E_x^1(\lambda, p)|\pi|,$$
$$\left\|V^{p,k} - \bar{V}^{p,k}\right\|^2_{\mathrm{L}^2(N)} \leq \lambda E_x^1(\lambda, p)|\pi|.$$

Under (H′), *there exists a version of* $Z^{p,k}$ *such that*

$$\left\|Z^{p,k} - \bar{Z}^{p,k}\right\|^2_{\mathrm{L}^2(W)} \leq \left(E_x^2(\lambda, p) + E_x^3(\lambda, p)k^2\right)|\pi|,$$

in which $E_x^i(\lambda, p), i \in \{1, 2, 3\}$ *have already been defined in* (4.31), (4.32) *and* (4.33) *respectively.*

Proof. This follows from Proposition 4.3.2, definitions of $\bar{Z}^{p,k}$ and $\bar{V}^{p,k}$ in (4.26) and straightforward computations. For any $n < N$, and $t_n \leq t \leq t_{n+1}$,

$$\mathbb{E}\left|V_t^{p,k} - \bar{V}_{t_n}^{p,k}\right|^2 \leq 2 \max_{n<N} \sup_{t_n \leq s \leq t_{n+1}} \mathbb{E}\left|V_t^{p,k} - V_{t_n}^{p,k}\right|^2 + 2\mathbb{E}\left|V_{t_n}^{p,k} - \bar{V}_{t_n}^{p,k}\right|^2.$$

By definition of $\bar{V}^{p,k}$ and Cauchy-Schwartz' inequality,

$$\begin{aligned}
\mathbb{E}\left|V_{t_n}^{p,k} - \bar{V}_{t_n}^{p,k}\right|^2 &= \mathbb{E}\left|V_{t_n}^{p,k} - \frac{1}{\Delta t_{n+1}}\mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} V_s^{p,k} ds\right]\right|^2 \\
&= \mathbb{E}\left|\frac{1}{\Delta t_{n+1}}\mathbb{E}_{t_n}\left[\int_{t_n}^{t_{n+1}} \left(V_{t_n}^{p,k} - V_s^{p,k}\right) ds\right]\right|^2 \\
&\leq \frac{1}{\Delta t_{n+1}}\mathbb{E}\int_{t_n}^{t_{n+1}} \left|V_{t_n}^{p,k} - V_s^{p,k}\right|^2 ds \\
&\leq \sup_{t_n \leq s \leq t_{n+1}} \mathbb{E}\left|V_s^{p,k} - V_{t_n}^{p,k}\right|^2,
\end{aligned}$$

so that with the result of Proposition 4.3.2, we get

$$\begin{aligned}
\mathbb{E}\left|V_t^{p,k} - \bar{V}_{t_n}^{p,k}\right|^2 &\leq 4 \max_{n<N} \sup_{t_n \leq t \leq t_{n+1}} \mathbb{E}\left|V_t^{p,k} - V_{t_n}^{p,k}\right|^2 \\
&\leq 4E_x^1(\lambda, p)\ |\pi|.
\end{aligned}$$

This allows to write

$$\begin{aligned}
\left\|V^{p,k} - \bar{V}^{p,k}\right\|^2_{\mathrm{L}^2(N)} &= \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \mathbb{E}\left|V_t^{p,k} - \bar{V}_{t_n}^{p,k}\right|^2 \lambda dt \\
&\leq \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \left(\max_{n<N} \sup_{t_n \leq t \leq t_{n+1}} \mathbb{E}\left|V_t^{p,k} - \bar{V}_{t_n}^{p,k}\right|^2\right) \lambda dt \\
&\leq 4T\lambda E_x^1(\lambda, p)\ |\pi|.
\end{aligned}$$

The same reasoning applied to $Z^{p,k}$ and $\bar{Z}^{p,k}$ allows to complete the proof. □

Proposition 4.3.1 and Corollary 4.3.1 lead finally to the expected bound on the regularity errors defined in (4.21).

Proposition 4.3.3. *Let assumption* (H) *be satisfied and* $|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right)$. *Then there exists some version of* V^p *such that*

$$\max_{n=1,\dots,N-1} \mathbb{E}\left[\sup_{t_n \le t \le t_{n+1}} \left|Y_t^p - Y_{t_n}^p\right|^2\right] \le E_x^1(\lambda,p)|\pi|$$
$$\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)} \le \lambda E_x^1(\lambda,p)|\pi|$$

Under (H′), *there exists some version of* Z^p *such that,*

$$\left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)} \le F_x^1(\lambda,p)|\pi|^{\frac{1}{2}},$$

where $E_x^i(\lambda,p), i \in \{1,2,3\}$ *have already been defined in* (4.31), (4.32) *and* (4.33) *respectively,* $R(\lambda,p)$ *in* (4.28) *and*

$$F_x^1(\lambda,p) := C\left(R(\lambda,p) + E_x^2(\lambda,p)\left(\frac{1}{\lambda p^2}\right)^{\frac{1}{2}} + E_x^3(\lambda,p)\right).$$

In particular, as $p \to +\infty$,

$$E_x^1(\lambda,p) = \mathcal{O}\left(\bar{C}_{x,\lambda}^{2,e}(1+\lambda)^2\lambda p^2 e^{C\lambda p^2}\right),$$
$$F_x^1(\lambda,p) = \mathcal{O}\left(\bar{C}_{x,\lambda}^{4,e}\lambda^5 p^6 e^{C\lambda p^2}\right).$$

recalling the definition of $\bar{C}_{x,\lambda}^{q,e}, q \in \{1,2\}$ *and* $\bar{C}_{\lambda}^{2,e}$ *in* (4.34) *and* (4.27).

Proof. The two first results are straightforward from the two first estimates of Corollary 4.3.1 which hold uniformly in k and the convergence of $(Y^{p,k}, V^{p,k})$ to (Y^p, V^p) in $\mathcal{S}^2 \times \mathrm{L}^2(N)$ (see Proposition 4.3.1). Finally, as $\left\|\bar{Z}^p - \bar{Z}^{p,k}\right\|^2_{\mathrm{L}^2(W)} \le \left\|Z^p - Z^{p,k}\right\|^2_{\mathrm{L}^2(W)}$, we have

$$\begin{aligned}\left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)} &\le 3\left\|Z^p - Z^{p,k}\right\|^2_{\mathrm{L}^2(W)} + 3\left\|Z^{p,k} - \bar{Z}^{p,k}\right\|^2_{\mathrm{L}^2(W)} + 3\left\|\bar{Z}^p - \bar{Z}^{p,k}\right\|^2_{\mathrm{L}^2(W)} \\ &\le 6\left\|Z^p - Z^{p,k}\right\|^2_{\mathrm{L}^2(W)} + 3\left\|Z^{p,k} - \bar{Z}^{p,k}\right\|^2_{\mathrm{L}^2(W)} \\ &\le 6R(\lambda,p)\frac{1}{k^2} + 3\left(E_x^2(\lambda,p) + E_x^3(\lambda,p)k^2\right)|\pi|,\end{aligned}$$

by Corollary 4.3.1 and Proposition 4.3.1. This hold for any regularization coefficient $k \ge 1$. Taking

$$k = |\pi|^{-\frac{1}{4}} \longrightarrow +\infty \quad \text{as } |\pi| \to 0$$

leads thus to

$$\left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)} \le 6R(\lambda,p)|\pi|^{\frac{1}{2}} + 3E_x^2(\lambda,p)|\pi| + 3E_x^3(\lambda,p)|\pi|^{\frac{1}{2}},$$

which allows to conclude since $|\pi| = \mathcal{O}(\frac{1}{\lambda p^2})$. □

4.4 Global convergence rate of the penalization approach

By propositions 4.2.1 and 4.3.3, we obtain the following explicit bound on the discretization error.

Corollary 4.4.1. *Assume* (H). *Then there exists a version of* V^p *such that, when* $|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right)$,

$$\mathcal{E}^\pi(Y^p) = \mathcal{O}_{p\to+\infty}\left((1+\lambda)^2\lambda p\bar{C}^{\lambda p^2}|\pi|^{\frac{1}{2}}\right),$$
$$\mathcal{E}^\pi(V^p) = \mathcal{O}_{p\to+\infty}\left((1+\lambda)^2\lambda^{\frac{3}{2}}p^2\bar{C}^{\lambda p^2}|\pi|^{\frac{1}{2}}\right).$$

Under (H′), *there exists a version of* Z^p *such that, when* $|\pi| = \mathcal{O}\left(\frac{1}{\lambda p^2}\right)$,

$$\mathcal{E}^\pi(Z^p) = \mathcal{O}_{p\to+\infty}\left((1+\lambda)^2\lambda^{\frac{5}{2}}p^3\bar{C}^{\lambda p^2}|\pi|^{\frac{1}{4}}\right),$$

for some constant $\bar{C}$ *greater than* 1 *which does not depend either on* λ, p *or* $|\pi|$.

Proof. The predominant term in both of bounds for $|\mathcal{E}^\pi(Y^p)|^2$ and $|\mathcal{E}^\pi(Y^p)|^2$ given by propositions 4.2.1 and 4.3.3 is the term involving $\left\|V^p - \bar{V}^p\right\|^2_{\mathrm{L}^2(N)}$. This implies

$$|\mathcal{E}^\pi(Y^p)|^2 = \mathcal{O}\left(\bar{C}^{2,e}_{x,\lambda}(1+\lambda)^2\lambda^2p^2e^{C\lambda p^2}|\pi|\right)$$
$$|\mathcal{E}^\pi(V^p)|^2 = \mathcal{O}\left(\bar{C}^{2,e}_{x,\lambda}(1+\lambda)^2(\lambda p^2)\lambda^2p^2e^{C\lambda p^2}|\pi|\right)$$

which, together with the definition of $\bar{C}^{2,e}_{x,\lambda}$ in (4.34) shows the two first estimations. Finally, the predominant term in the bound for $|\mathcal{E}^\pi(V^p)|^2$ given by propositions 4.2.1 and 4.3.3 is the term involving $\left\|Z^p - \bar{Z}^p\right\|^2_{\mathrm{L}^2(W)}$. This leads to

$$|\mathcal{E}^\pi(Z^p)|^2 = \mathcal{O}\left(\bar{C}^{4,e}_{x,\lambda}\lambda^5p^6e^{C\lambda p^2}|\pi|^{\frac{1}{2}}\right),$$

which, together with the definition of $\bar{C}^{4,e}_{x,\lambda}$ in (4.34) allows to conclude. □

Finally, we sum up in Theorem 4.4.1 the global bound that we obtain on the error made when solving backward scheme (4.5) for solving impulse control problem (1.2) via an approach by penalization. This is a direct consequence from Theorem 3.2.1 and Corollary 4.4.1 and we present in particular the following relevant error made on the value process:

$$\mathcal{E}^p + \mathcal{E}^\pi(Y^p) = \sup_{0\le t\le T}\left|v(t,x) - Y^{p,t,x}_t\right| + \left(\sup_{0\le t\le T}\mathbb{E}\left|Y^p_t - \bar{Y}^{p,\pi}_t\right|^2\right)^{\frac{1}{2}}.$$

Theorem 4.4.1. *Let the assumptions of Theorem 3.2.1 be satisfied. Then*

$$\mathcal{E}^p + \mathcal{E}^\pi(Y^p) = \mathcal{O}\left(\bar{n}\bar{C}^{\bar{n}}\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}} + (1+\lambda)^2\lambda p\bar{C}^{\lambda p^2}|\pi|^{\frac{1}{2}}\right), \quad \forall\alpha\in\left(0,\frac{1}{2}\right),$$

for some constant $\bar{C}$ *greater than* 1 *which does not depend either on* λ, p, $|\pi|$ *or* α. *Thus, for a sufficiently small time step* $|\pi|$ *with respect to* λ *and* p, *the global error is such that*

$$[\mathcal{E}^p + \mathcal{E}^\pi(Y^p)]^* \quad = \mathcal{O}_{p\to+\infty}\left(\frac{1}{(\lambda p)^{\frac{1}{2}-\alpha}}\right), \quad \forall\alpha\in\left(0,\tfrac{1}{2}\right).$$

4.A Appendix

These results follow from same arguments as the one used by Elie [48], but were carefully performed by taking into account the dependence in the penalization parameter p, the jump intensity λ and the regularization coefficient k.

4.A.1 A priori estimates

Lemma 4.A.1. Assume (H). Then, for any $0 \leq s \leq t \leq T$:

$$\mathbb{E}\left[\sup_{s\leq u\leq t}|Y_u^p - Y_s^p|^2\right] \leq C\left(\bar{C}_{x,\lambda}^{2,e}(1+\lambda p)^2|t-s|^2 + \|Z^p\|^2_{\mathrm{L}^2_{[s,t]}(W)} + \left(1+\lambda p^2|t-s|\right)\|V^p\|^2_{\mathrm{L}^2_{[s,t]}(N)}\right),$$

in which $\bar{C}_{x,\lambda}^{2,e}$ is defined in (4.27).

Proof. Let $0 \leq s \leq t \leq T$. We have

$$Y_u^p - Y_s^p = -\int_s^u f^p(X_r, V_r^p)dr + \int_s^u Z_r^p dW_r + \int_s^u V_r^p d\tilde{N}_r.$$

With Doob's and Jensen's inequalities, we have

$$\mathbb{E}\left[\sup_{s\leq u\leq t}|Y_u^p - Y_s^p|^2\right] \leq C\left(|t-s|\,\mathbb{E}\left[\int_s^t |f^p(X_r,V_r^p)|^2\,dr\right] + \|Z^p\|^2_{\mathrm{L}^2_{[s,t]}(W)} + \|V^p\|^2_{\mathrm{L}^2_{[s,t]}(N)}\right).$$

Lipschitz property of f^p (see Lemma 4.2.3) and Lemma 4.2.1 leads to

$$\begin{aligned}\mathbb{E}\left[\int_s^t |f^p(X_r,V_r^p)|^2\,dr\right] &\leq C(1+\lambda p)^2|t-s| + C(1+\lambda p)^2|t-s|\,\|X\|^2_{\mathcal{S}^2} + C\lambda p^2\,\|V^p\|^2_{\mathrm{L}^2_{[s,t]}(N)}\\ &\leq \bar{C}_{x,\lambda}^{2,e}(1+\lambda p)^2|t-s| + C\lambda p^2\,\|V^p\|^2_{\mathrm{L}^2_{[s,t]}(N)},\end{aligned}$$

which allows to complete the estimation of $\mathbb{E}\left[\sup_{s\leq u\leq t}|Y_u^p - Y_s^p|^2\right]$. □

Lemma 4.A.2. Assume (H). Then

$$\|Y^p\|^2_{\mathcal{S}^2} + \|Z^p\|^2_{\mathrm{L}^2(W)} + \|V^p\|^2_{\mathrm{L}^2(N)} \leq E(\lambda,p), \tag{4.35}$$

where

$$E(\lambda,p) := e^{C\lambda p^2}\left(\lambda p^2 + \left(1+\frac{1}{\lambda}\right)(1+\lambda p)^2\right).$$

Proof. Divide $[0,T]$ in $K \geq 1$ intervals $([s_l, s_{l+1}])_{l=0,\dots,K-1}$ with same length $\delta_K = \frac{T}{K}$. (4.2) gives, for any $s_l \leq s \leq s_{l+1}$,

$$Y_s^p + \int_s^{s_{l+1}} Z_r^p dW_r + \int_s^{s_{l+1}} V_r^p d\tilde{N}_r = Y_{s_{l+1}}^p + \int_s^{s_{l+1}} f^p(X_r,V_r^p)dr, \tag{4.36}$$

so that

$$|Y_s^p| \leq \mathbb{E}\left[\left|Y_{s_{l+1}}^p\right| + \int_{s_l}^{s_{l+1}} |f^p(X_r,V_r^p)|\,dr|\mathcal{F}_s\right].$$

By Jensen's and Doob's inequalities, this implies that

$$\mathbb{E}\left[\sup_{s_l\le s\le s_{l+1}}|Y_s^p|^2\right]\le C\mathbb{E}\left[\left|Y_{s_{l+1}}^p\right|^2+\left|\int_{s_l}^{s_{l+1}}|f^p(X_r,V_r^p)|\,dr\right|^2\right],$$

so that (4.36) gives

$$\|Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}+\|Z^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)}+\|V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}\le C\mathbb{E}\left[\left|Y_{s_{l+1}}^p\right|^2+\left|\int_{s_l}^{s_{l+1}}|f^p(X_r,V_r^p)|\,dr\right|^2\right].$$

By Cauchy-Schwartz's inequality and Lipschitz property of f^p (see Lemma 4.2.3),

$$\begin{aligned}&\|Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}+\|Z^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)}+\|V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}\\ \le\;& C\mathbb{E}\left|Y_{s_{l+1}}^p\right|^2+C\delta_K^3(1+\lambda p)^2+C\delta_K^2(1+\lambda p)^2\|X\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}+C\delta_K\lambda p^2\|V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}.\end{aligned}$$

When choosing $\delta_K=\frac{1}{2C\lambda p^2}$, we get

$$\begin{aligned}&\|Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}+\|Z^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)}+\frac{1}{2}\|V^p\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}\\ \le\;& C\mathbb{E}\left|Y_{s_{l+1}}^p\right|^2+C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\|X\|^2_{\mathcal{S}^2}.\end{aligned}\tag{4.37}$$

Iterating on s_l until $s_K=T$ in equation (4.37) gives for any $l=0,\dots,K-1$,

$$\begin{aligned}\|Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}&\le C^{K-l}\mathbb{E}\,|g(X_T)|^2+(K-l)C^{K-l}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\|X\|^2_{\mathcal{S}^2}\\ &\le C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right)\|X\|^2_{\mathcal{S}^2}\le\bar{C}^{2,e}_{x,\lambda}C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right),\end{aligned}$$

by Lipschitz property of g and Lemma 4.2.1. This implies (recall $K=2CT\lambda p^2$),

$$\|Y^p\|^2_{\mathcal{S}^2}\le K\max_{l=0,\dots,K-1}\|Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}\le\bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2+(1+\lambda p)^2\right).$$

Summing up over $l=0,\dots,K-1$ in equation (4.37) leads finally to

$$\begin{aligned}\|Z^p\|^2_{\mathrm{L}^2(W)}+\frac{1}{2}\|V^p\|^2_{\mathrm{L}^2(N)}&\le C\sum_{l=0}^{K-1}\mathbb{E}\left|Y_{s_{l+1}}^p\right|^2+C\frac{(1+\lambda p)^2}{\lambda p^2}\|X\|^2_{\mathcal{S}^2}\\ &\le CK\max_{l=0,\dots,K-1}\|Y^p\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}+\bar{C}^{2,e}_{x,\lambda}\frac{(1+\lambda p)^2}{\lambda p^2}\\ &\le\bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2+\left(1+\frac{1}{\lambda}\right)(1+\lambda p)^2\right),\end{aligned}$$

which concludes the proof.

□

4.A.2 Proof of Proposition 4.3.2

For ease of notations, we consider the case when $d=1$ in the following proofs. The case $d\ge1$ can be handled with same arguments and in particular, it does not impact the final rates of convergence, since the properties of the regularized FBSDE coefficients in Lemma 4.3.1 hold in $\mathbb{R}^d$.

Part 1: Estimate for $Y^{p,k}$.

By Lemma 4.A.1, we already have, for any $n = 0, \dots, N-1$,

$$\mathbb{E}\left[\sup_{t_n \le t \le t_{n+1}} \left|Y_t^{p,k} - Y_{t_n}^{p,k}\right|^2\right] \le C\left[\bar{C}_{x,\lambda}^{2,e}(1+\lambda p)^2|\pi|^2 + \left\|Z^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)} + \left(1+\lambda p^2|\pi|\right)\left\|V^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)}\right] \tag{4.38}$$

In the sequel, we provide upper bounds of order $|\pi|$ on $\left\|Z^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)}$ and $\left\|V^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)}$.

Step 1. We get a representation to $Z^{p,k}$ by considering the Malliavin derivative of solution $Y^{p,k}$ to (4.24). For any $0 < s \le t \le T$,

$$D_s Y_t^{p,k} = Z_s^{p,k} - \int_s^t \nabla f^{p,k}(X_r, V_r^{p,k}) \cdot D_s\left(X_r, V_r^{p,k}\right) dr + \int_s^t D_s Z_r^{p,k} dW_r + \int_s^t D_s V_r^{p,k} d\tilde{N}_r,$$

so that after possibly passing to a suitable version:

$$Z_t^{p,k} = D_t Y_t^{p,k} = \Upsilon_t^{p,k,t},$$

where $(\Upsilon_t^{p,k,s}, \zeta_t^{p,k,s}, \Psi_t^{p,k,s})_{0<s\le t\le T}$ is the unique solution to FBSDE

$$\begin{cases} \Upsilon_t^{p,k,s} = \nabla g^k(X_T^k)\chi_T^{k,s} + \int_t^T \nabla f^{p,k}(X_r^k, V_r^{p,k}) \cdot \left(\chi_r^{k,s}, \Psi_r^{p,k,s}\right) dr - \int_t^T \zeta_r^{p,k,s} dW_r - \int_t^T \Psi_r^{p,k,s} d\tilde{N}_r \\ \chi_t^{k,s} = \sigma^k(X_{s^-}^k) + \int_s^t \nabla b^k(X_r^k)\chi_r^{k,s} dr + \int_s^t \nabla \sigma^k(X_r^k)\chi_r^{k,s} dW_r + \int_s^t \nabla \gamma^k(X_{r^-}^k)\chi_{r^-}^{k,s} dN_r. \end{cases} \tag{4.39}$$

Up to a suitable version, we also have $D_s X^k = \chi^{k,s}$. Let us show that

$$\sup_{0\le s\le T}\left\{\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} + \left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}\right\} \tag{4.40}$$

is bounded by some constant depending on (λ, p) to be determined explicitly. As the coefficients of X^k verify condition (H) and belong to $\mathcal{C}_b^1$, we have by Jensen's and Doob's inequalities, $\forall 0 \le s \le t \le T$,

$$\begin{aligned} \mathbb{E}\left[\sup_{s\le r\le t}\left|\chi_r^{k,s}\right|^2\right] &\le 4\mathbb{E}\left|\sigma^k(X_{s^-}^k)\right|^2 + 4T\mathbb{E}\int_s^t \left|\nabla b^k(X_r^k)\chi_r^{k,s} + \nabla\gamma^k(X_{r^-}^k)\chi_{r^-}^{k,s}\lambda\right|^2 dr \\ &\quad + 4\mathbb{E}\int_s^t \left|\nabla\sigma^k(X_r^k)\chi_r^{k,s}\right|^2 dr + 4\mathbb{E}\int_s^t \left|\nabla\gamma^k(X_{r^-}^k)\chi_{r^-}^{k,s}\right|^2 \lambda dr \\ &\le C\left(1 + \mathbb{E}\left|X_{s^-}^k\right|^2\right) + C(1+\lambda+\lambda^2)\int_s^t \mathbb{E}\left|\chi_r^{k,s}\right|^2 dr \\ &\le \bar{C}_{x,\lambda}^{2,e} + C(1+\lambda+\lambda^2)\int_s^t \mathbb{E}\left[\sup_{s\le u\le r}\left|\chi_u^{k,s}\right|^2\right] dr. \end{aligned}$$

because Lemma 4.2.1 also holds for X^k. Applying Gronwall's lemma to $t \to \mathbb{E}\left[\sup_{s\le r\le t}\left|\chi_r^{k,s}\right|^2\right]$, we get

$$\mathbb{E}\left[\sup_{s\le r\le t}\left|\chi_r^{k,s}\right|^2\right] \le \bar{C}_{x,\lambda}^{2,e} e^{C(1+\lambda+\lambda^2)(t-s)},$$

so that

$$\sup_{0\le s\le T}\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} \le C(1+x^2)(1+\lambda)^2 e^{C(1+\lambda)^2} = \bar{C}^{2,e}_{x,\lambda}. \tag{4.41}$$

Divide now $[s,T]$ in $K\ge 1$ intervals $([s_l,s_{l+1}])_{l=0,\dots,K-1}$ with same length $\delta_K = \frac{T-s}{K} \le \frac{T}{K}$. For $s_l \le t \le s_{l+1}$, we have

$$\Upsilon^{p,k,s}_t + \int_t^{s_{l+1}} \zeta^{p,k,s}_r dW_r + \int_t^{s_{l+1}} \Psi^{p,k,s}_r d\tilde{N}_r = \Upsilon^{p,k,s}_{s_{l+1}} + \int_t^{s_{l+1}} \nabla f^{p,k}(X^k_r, V^{p,k}_r)\cdot\left(\chi^{k,s}_r, \Psi^{p,k,s}_r\right) dr$$

which leads with the same arguments as in the proof of Lemma 4.A.2 to

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\le\ & C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^2 + C\mathbb{E}\left[\left|\int_{s_l}^{s_{l+1}} \left|\nabla f^{p,k}(X^k_r, V^{p,k}_r)\cdot\left(\chi^{k,s}_r, \Psi^{p,k,s}_r\right)\right| dr\right|^2\right].
\end{aligned}$$

By Lemma 4.3.1-(v), we have

$$\left|\nabla f^{p,k}(X^k_r, V^{p,k}_r)\cdot\left(\chi^{k,s}_r, \Psi^{p,k,s}_r\right)\right| \le C\left((1+\lambda p)\left|\chi^{k,s}_r\right| + \lambda p\left|\Psi^{p,k,s}_r\right|\right)$$

so that by Jensen's inequality,

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\le\ & C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^2 + C\delta_K\mathbb{E}\left[\int_{s_l}^{s_{l+1}} \left|\nabla f^{p,k}(X^k_r, V^{p,k}_r)\cdot\left(\chi^{k,s}_r, \Psi^{p,k,s}_r\right)\right|^2 dr\right] \\
\le\ & C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^2 + C\delta_k^2(1+\lambda p)^2\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} + C\delta_K\lambda p^2\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}.
\end{aligned}$$

Choosing $\delta_K = \frac{1}{2C\lambda p^2}$, we get

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\le\ & C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^2 + C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}.
\end{aligned} \tag{4.42}$$

Iterating on s_l until $s_K = T$ in previous equation gives for all $l = 0,\dots,K-1$,

$$\begin{aligned}
\left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} &\le C^{K-k}\mathbb{E}\left|\nabla g^k(X^k_T)\chi^{k,s}_T\right|^2 + (K-k)C^{K-k}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} \\
&\le C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right)\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} \le \bar{C}^{2,e}_{x,\lambda}C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right),
\end{aligned}$$

by Lemma 4.3.1-(iv) for ∇g^k and estimate (4.41). This leads to (recall $K \le 2C\lambda p^2$)

$$\sup_{0\le s\le T}\left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} \le K\max_{l=0,\dots,K-1}\left\|\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \le \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right).$$

In conclusion, we have

$$\left\|Z^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)} = \mathbb{E}\int_{t_n}^{t_{n+1}} \left|Z_s^{p,k}\right|^2 ds \leq \left(\sup_{0\leq t\leq T} \mathbb{E}\left|\Upsilon_t^{p,k,t}\right|^2\right)|\pi| \leq \bar{C}^{2,e}_{x,\lambda} e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|\pi|.$$

Step 2. As the coefficients of FBSDE (4.25)-(4.24) belong to $\mathcal{C}_b^1$, there exists a version of $V^{p,k}$ such that

$$V_t^{p,k} = Y_t^{p,k}(t, x_t^{k,1}) - Y_t^{p,k}(t, x_t^{k,2}), \quad \forall 0 < t \leq T,$$

where $x_t^{k,1} = X_{t^-}^k + \gamma^k(X_{t^-}^k) \in \mathrm{L}^2(\mathcal{F}_t)$, $x_t^{k,2} = X_{t^-}^k \in \mathrm{L}^2(\mathcal{F}_t)$. $Y^{p,k}(t,x)$ denotes the value process of the solution $(Y_s^{p,k}(t,x), Z_s^{p,k}(t,x), V_s^{p,k}(t,x))_{t\leq s\leq T}$ to BSDE (4.24), when the state variable X^k is replaced by $X^k(t,x)$ which is the solution to SDE (4.25) on $[t,T]$ with initial condition $X_t^k(t,x) = x$. Denote for ease of notation:

$$\begin{array}{rl} & \delta X^k = X^k(t, x_t^{k,1}) - X^k(t, x_t^{k,2}) \\ \forall A \in \{Y, Z, V\}, & \delta A^{p,k} = A^{p,k}(t, x_t^{k,1}) - A^{p,k}(t, x_t^{k,2}) \\ \forall \xi \in \{b, \sigma, \gamma, \kappa, g\}, & \delta \xi_\cdot^k = \xi^k(X_\cdot^k(t, x_t^{k,1})) - \xi^k(X_\cdot^k(t, x_t^{k,2})) \\ & \delta f_\cdot^{p,k} = f^{p,k}(X_\cdot^k(t, x_t^{k,1}), V_\cdot^{p,k}(t, x_t^{k,1})) - f^{p,k}(X_\cdot^k(t, x_t^{k,2}), V_\cdot^{p,k}(t, x_t^{k,2})). \end{array}$$

First, let us show that

$$\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq \bar{C}^{2,e}_{x,\lambda}.$$

We have $\forall t \leq s \leq T$,

$$\delta X_s^k = x_t^{k,1} - x_t^{k,2} + \int_t^s \left[\delta b_r^k + \delta\gamma_r^k \lambda\right] dr + \int_t^s \delta\sigma_r^k dW_r + \int_t^s \delta\gamma_r^k d\tilde{N}_r.$$

By Jensen's and Doob's inequalities, we get $\forall t \leq u \leq T$,

$$\begin{aligned} & \mathbb{E}\left[\sup_{t\leq s\leq u}\left|\delta X_s^k\right|^2\right] \\ \leq\ & 4\mathbb{E}\left|x_t^{k,1} - x_t^{k,2}\right|^2 + 4T\mathbb{E}\int_t^u \left|\delta b_r^k + \delta\gamma_r^k\lambda\right|^2 dr + 4\mathbb{E}\int_t^u \left|\delta\sigma_r^k\right|^2 dr + 4\mathbb{E}\int_t^u \left|\delta\gamma_r^k\right|^2 \lambda dr \\ \leq\ & C\left(1 + \mathbb{E}\left|X_{t^-}^k\right|^2\right) + C\left(1+\lambda+\lambda^2\right)\int_t^u \mathbb{E}\left|\delta X_s^k\right|^2 ds \\ \leq\ & \bar{C}^{2,e}_{x,\lambda} + C\left(1+\lambda+\lambda^2\right)\int_t^u \mathbb{E}\left[\sup_{t\leq r\leq s}\left|\delta X_r^k\right|^2\right] ds, \end{aligned}$$

because Lemma 4.2.1 holds for X^k. By Gronwall's lemma, this implies

$$\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq C(1+x^2)(1+\lambda)^2 e^{C(1+\lambda)^2} = \bar{C}^{2,e}_{x,\lambda}. \tag{4.43}$$

Let us show that

$$\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq \bar{C}^{2,e}_{x,\lambda} e^{C\lambda p^2}(1+\lambda p)^2.$$

We have, $\forall t \leq s \leq T$,

$$\delta Y_s^{p,k} = \delta g_T^k + \int_s^T \delta f_r^{p,k} dr - \int_s^T \delta Z_r^{p,k} dW_r - \int_s^T \delta V_r^{p,k} d\tilde{N}_r. \tag{4.44}$$

With the same idea as in the proof of Lemma 4.A.2, divide $[t,T]$ in $K \geq 1$ intervals $([s_l, s_{l+1}])_{l=0,\dots,K-1}$ with same length $\delta_K = \frac{T-t}{K} \leq \frac{T}{K}$. (4.44) gives, for any $s_l \leq s \leq s_{l+1}$,

$$\delta Y_s^{p,k} + \int_s^{s_{l+1}} \delta Z_r^{p,k} dW_r + \int_s^{s_{l+1}} \delta V_r^{p,k} d\tilde{N}_r = \delta Y_{s_{l+1}}^{p,k} + \int_s^{s_{l+1}} \delta f_r^{p,k} dr.$$

Burkholder-Davis-Gundy's inequality implies thus

$$\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta Z^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \leq C\mathbb{E}\left[\left|\delta Y_{s_{l+1}}^{p,k}\right|^2 + \left|\int_{s_l}^{s_{l+1}} \left|\delta f_r^{p,k}\right| dr\right|^2\right].$$

By Lipschitz properties of $f^{p,k}$ (see Lemma 4.3.1-(iii)),

$$\left|\delta f_r^{p,k}\right| \leq C\left((1+\lambda p)\left|\delta X_r^k\right| + \lambda p \left|\delta V_r^{p,k}\right|\right)$$

so that by Cauchy-Schwartz's inequality, we get

$$\begin{aligned}
&\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta Z^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\leq\ & C\mathbb{E}\left|\delta Y_{s_{l+1}}^{p,k}\right|^2 + C\delta_K^2(1+\lambda p)^2 \left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + C\delta_K \lambda p^2 \left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}.
\end{aligned}$$

Choosing $\delta_K = \frac{1}{2C\lambda p^2}$ and iterating on s_l until $s_K = T$, we get

$$\begin{aligned}
&\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta Z^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\leq\ & C\mathbb{E}\left|\delta Y_{s_{l+1}}^{p,k}\right|^2 + C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \\
\leq\ & C^{K-k}\mathbb{E}\left|g^k(X_T^k(t,x_t^{k,1})) - g^k(X_T^k(t,x_t^{k,2}))\right|^2 + (K-k)C^{K-k}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \\
\leq\ & C^K\mathbb{E}\left|\delta X_T^k\right|^2 + KC^K\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \\
\leq\ & C^K\left(1 + \frac{(1+\lambda p)^2}{\lambda p^2}\right)\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq \bar{C}^{2,e}_{x,\lambda} C^K\left(1 + \frac{(1+\lambda p)^2}{\lambda p^2}\right)
\end{aligned}$$

by Lipschitz property of g^k and estimate (4.43), leading to (recall $K \leq 2C\lambda p^2$)

$$\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq K \max_{l=0,\dots,K-1}\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \leq \bar{C}^{2,e}_{x,\lambda} e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right).$$

By arbitrariness of $t \in [0,T]$, we have thus

$$\begin{aligned}
\left\|V^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)} = \mathbb{E}\int_{t_n}^{t_{n+1}}\left|V_s^{p,k}\right|^2 \lambda ds &\leq \left(\sup_{0\leq t\leq T}\mathbb{E}\left|\delta Y_t^{p,k}\right|^2\right)\lambda|\pi| \\
&\leq \lambda \bar{C}^{2,e}_{x,\lambda} e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|\pi|.
\end{aligned}$$

Injecting the estimates for $\left\|Z^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(W)}$ and $\left\|V^{p,k}\right\|^2_{\mathrm{L}^2_{[t_n,t_{n+1}]}(N)}$ in (4.38), leads to

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t_n\le t\le t_{n+1}}\left|Y_t^{p,k}-Y_{t_n}^{p,k}\right|^2\right]\\
\le\ & \bar{C}^{2,e}_{x,\lambda}\left[(1+\lambda p)^2|\pi|+e^{C\lambda p^2}\left(\lambda p^2+(1+\lambda p)^2\right)+\lambda\left(1+\lambda p^2|\pi|\right)e^{C\lambda p^2}\left(\lambda p^2+(1+\lambda p)^2\right)\right]|\pi|\\
\le\ & E^1_x(\lambda,p)|\pi|
\end{aligned}$$

since $|\pi|=\mathcal{O}\left(\frac{1}{\lambda p^2}\right)$, recalling (4.31).

Part 2: Estimate for $V^{p,k}$.

As the coefficients of FBSDE (4.25)-(4.24) belong to $\mathcal{C}^1_b$, there exists a version of $V^{p,k}$ such that $\forall t\in[0,T]$:

$$V_t^{p,k}=Y_t^{p,k}(t,X_{t^-}^{p,k}+\gamma^k(t,X_{t^-}^{p,k}))-Y_t^{p,k}(t,X_{t^-}^{p,k}),$$

where $Y^{p,k}(t,x)$ denotes the value process solution to BSDE (4.24), where the state X^k is replaced by $X^k(t,x)$, solution to (4.25) on $[t,T]$ with initial condition $X^k_t(t,x)=x$. In consequence, $\forall 0<s\le t\le T$,

$$\begin{aligned}
&\sup_{s\le r\le t}\mathbb{E}\left|V_r^{p,k}-V_s^{p,k}\right|^2\\
=\ &\sup_{s\le r\le t}\mathbb{E}\left|\left(Y_r^{p,k}(r,x_r^{k,1})-Y_r^{p,k}(r,x_r^{k,2})\right)-\left(Y_s^{p,k}(s,x_s^{k,1})-Y_s^{p,k}(s,x_s^{k,2})\right)\right|^2\\
\le\ & 2\sup_{s\le r\le t}\mathbb{E}\left|Y_r^{p,k}(r,x_r^{k,1})-Y_s^{p,k}(s,x_s^{k,1})\right|^2+2\sup_{s\le r\le t}\mathbb{E}\left|Y_r^{p,k}(r,x_r^{k,2})-Y_s^{p,k}(s,x_s^{k,2})\right|^2
\end{aligned}$$

where (we adopt the same kind of notation as in Part 1) for any $r>0$, $x_r^{k,1}:=X^k_{r^-}+\gamma^k(X^k_{r^-})$, $x_r^{k,2}:=X^k_{r^-}$. Let us show that the first part of the right-hand side of previous inequality is such that

$$e_1:=\sup_{s\le r\le t}\mathbb{E}\left|Y_r^{p,k}(r,x_r^{k,1})-Y_s^{p,k}(s,x_s^{k,1})\right|^2\le E^1_x(\lambda,p)|t-s|.$$

As the same estimate holds for the second part, we will be able to conclude. Decompose

$$e_1\le 2\mathbb{E}\left[\sup_{s\le r\le t}\left|Y_r^{p,k}(s,x_s^{k,1})-Y_s^{p,k}(s,x_s^{k,1})\right|^2\right]+2\sup_{s\le r\le t}\mathbb{E}\left|Y_r^{p,k}(r,x_r^{k,1})-Y_r^{p,k}(s,x_s^{k,1})\right|^2$$

and denote by $(\bar{Y}_r^{p,k})_{s\le r\le T}=(Y_r^{p,k}(s,x_s^{k,1}))_{s\le r\le T}$. It is easy to prove, by the same arguments that we used in Part 1 (with Lemma 4.A.1), that

$$\mathbb{E}\left[\sup_{s\le r\le t}\left|\bar{Y}_r^{p,k}-\bar{Y}_s^{p,k}\right|^2\right]\le E^1_x(\lambda,p)|t-s|.$$

Indeed, set again

$$\forall A\in\{X,Y^p,Z^p,V^p\},\delta A^k=A^k(r,x_r^{k,1})-A^k(s,x_s^{k,1})$$

and let us prove that

$$\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[r,T]}} \leq \bar{C}^{2,e}_{x,\lambda}|r-s|. \tag{4.45}$$

With the same kind of arguments that we already used, we have, $\forall s \leq r \leq u \leq T$,

$$\mathbb{E}\left[\sup_{r\leq t\leq u}\left|\delta X^k_t\right|^2\right] \leq 4\mathbb{E}\left|x^{k,1}_r - x^{k,1}_s\right|^2 + C(1+\lambda+\lambda^2)|r-s| + C\int_s^r \mathbb{E}\left[\sup_{s\leq t\leq r}\left|X^k_t(s,x^{k,1}_s)\right|^2\right]dt + C(1+\lambda+\lambda^2)\int_r^u \mathbb{E}\left[\sup_{r\leq s\leq t}\left|\delta X^k_s\right|^2\right]dt.$$

As the coefficients of jump diffusion X^k satisfy (H_X), Lemma 4.2.1 implies

$$\left\|X^k\right\|_{S^2} + \left\|X^k(s,x^{k,1})\right\|_{S^2_{[s,T]}} \leq \bar{C}^{2,e}_{x,\lambda}.$$

Besides,

$$\mathbb{E}\left|x^{k,1}_r - x^{k,1}_s\right|^2 = \mathbb{E}\left|X^k_{r^-} + \gamma^k(X^k_{r^-}) - \left[X^k_{s^-} + \gamma^k(X^k_{s^-})\right]\right|^2 \leq C\mathbb{E}\left|X^k_{r^-} - X^k_{s^-}\right|^2$$

and again with Doob's and Jensen's inequalities,

$$\begin{aligned}
&\mathbb{E}\left|X^k_{r^-} - X^k_{s^-}\right|^2 \\
\leq\ & C\mathbb{E}\int_s^r \left|b^k(X^k_u) + \gamma^k(X^k_u)\lambda\right|^2 du + C\mathbb{E}\int_s^r \left|\sigma^k(X^k_u)\right|^2 du + C\mathbb{E}\int_s^r \left|\gamma^k(X^k_u)\right|^2 \lambda du \\
\leq\ & C(1+\lambda+\lambda^2)|r-s| + C\int_s^r \mathbb{E}\left[\sup_{0\leq u\leq T}\left|X^k_u\right|^2\right]du.
\end{aligned}$$

Thus

$$\mathbb{E}\left|x^{k,1}_r - x^{k,1}_s\right|^2 \leq C\mathbb{E}\left|X^k_{r^-} - X^k_{s^-}\right|^2 \leq C(1+\lambda+\lambda^2)|r-s| + \bar{C}^{2,e}_{x,\lambda}|r-s| \leq \bar{C}^{2,e}_{x,\lambda}|r-s| \tag{4.46}$$

and

$$\mathbb{E}\left[\sup_{r\leq t\leq u}\left|\delta X^k_s\right|^2\right] \leq \bar{C}^{2,e}_{x,\lambda}|r-s| + C(1+\lambda+\lambda^2)\int_r^u \mathbb{E}\left[\sup_{r\leq s\leq t}\left|\delta X^k_s\right|^2\right]dt,$$

which implies, by Gronwall's lemma,

$$\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[r,T]}} \leq \bar{C}^{2,e}_{x,\lambda}e^{C(1+\lambda+\lambda^2)T}|r-s| = \bar{C}^{2,e}_{x,\lambda}|r-s|.$$

It remains us to estimate

$$\sup_{s\leq r\leq t}\mathbb{E}\left|Y^{p,k}_r(r,x^{k,1}_r) - Y^{p,k}_r(s,x^{k,1}_s)\right|^2 \leq \sup_{s\leq r\leq t}\mathbb{E}\left[\sup_{r\leq u\leq T}\left|Y^{p,k}_u(r,x^{k,1}_r) - Y^{p,k}_u(s,x^{k,1}_s)\right|^2\right].$$

With the same ideas as in Part 1 and same shorthand notations, $\forall s \leq r \leq u \leq T$,

$$\delta Y^{p,k}_u = \delta g^k_T + \int_u^T \delta f^{p,k}_r dr - \int_u^T \delta Z^{p,k}_r dW_r - \int_u^T \delta V^{p,k}_r d\tilde{N}_r.$$

By dividing $[r,T]$ in $K \geq 1$ sub-intervals $([s_l, s_{l+1}])_{l=0,\dots,K-1}$ with same length $\delta_K = \frac{T-r}{K} \leq \frac{T}{K}$, we first have

$$\begin{aligned} &\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta Z^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq\ & C\mathbb{E}\left|\delta Y^{p,k}_{s_{l+1}}\right|^2 + C\delta_K^2(1+\lambda p)^2\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + C\delta_K\lambda p^2\left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}. \end{aligned}$$

Choosing $\delta_K = \frac{1}{2C\lambda p^2}$, iterating on s_l until $s_K = T$, together with estimate (4.45) for $\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[r,T]}}$ and Lipschitz property of g^k,

$$\begin{aligned} &\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta Z^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\delta V^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq\ & C^{K-k}\mathbb{E}\left|g^k(X^k_T(r,x^{k,1}_r)) - g^k(X^k_T(s,x^{k,1}_s))\right|^2 + (K-k)C^{K-k}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[r,T]}} \\ \leq\ & C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right)\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[r,T]}} \\ \leq\ & \bar{C}^{2,e}_{x,\lambda}C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right)|r-s|. \end{aligned}$$

With $K \leq 2C\lambda p^2$, this leads to

$$\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[r,T]}} \leq K \max_{l=0,\dots,K-1}\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \leq \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|r-s|.$$

Finally,

$$\begin{aligned} \sup_{s\leq r\leq t}\mathbb{E}\left|Y^{p,k}_r(r,x^{k,1}_r) - Y^{p,k}_r(s,x^{k,1}_s)\right|^2 &\leq \sup_{s\leq r\leq t}\left\|\delta Y^{p,k}\right\|^2_{\mathcal{S}^2_{[r,T]}} \\ &\leq \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|t-s|, \end{aligned}$$

leading simply to $e_1 \leq E^1_x(\lambda,p)|t-s|$, which allows to conclude.

Part 3: Estimate for $Z^{p,k}$.

Recall the representation given for $Z^{p,k}$ in Part 1:

$$Z^{p,k}_t = \Upsilon^{p,k,t}_t,$$

where $(\Upsilon^{p,k,s}_t, \zeta^{p,k,s}_t, \Psi^{p,k,s}_t)_{0<s\leq t\leq T}$ is the unique solution to FBSDE (4.39). We write then $\forall s \leq t \leq T$:

$$\mathbb{E}\left|Z^{p,k}_t - Z^{p,k}_s\right|^2 = \mathbb{E}\left|\Upsilon^{p,k,t}_t - \Upsilon^{p,k,s}_s\right|^2 \leq 2\underbrace{\mathbb{E}\left|\Upsilon^{p,k,s}_t - \Upsilon^{p,k,s}_s\right|^2}_{:=z_1} + 2\underbrace{\mathbb{E}\left|\Upsilon^{p,k,t}_t - \Upsilon^{p,k,s}_t\right|^2}_{:=z_2}.$$

Step 1. To control part z_1, we use the same kind of arguments that we used in Part 1. We have already (see (4.41) in Part 1)

$$\sup_{0\leq s\leq T}\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} \leq \bar{C}^{2,e}_{x,\lambda}.$$

Applying Lemma 4.A.1 to $(\Upsilon^{p,k,s}, \Psi^{p,k,s}, \zeta^{p,k,s})$ together with Lemma 4.3.1-(v) leads similarly to

$$z_1 \leq C\left[\bar{C}^{2,e}_{x,\lambda}(1+\lambda p)^2|t-s|^2 + \left\|\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,t]}(W)} + \left(1+\lambda p^2|t-s|\right)\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,t]}(N)}\right].$$

Let us first show that

$$\sup_{0\leq s\leq T}\left\|\zeta^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}$$

is bounded by some constant depending on (λ, p) to be determined explicitly. We get a representation to $\zeta^{p,k,s}$ by considering the Malliavin derivative of solution $\Upsilon^{p,k,s}$ to (4.39). For any $0 < s \leq u \leq t \leq T$,

$$\begin{aligned} D_u\Upsilon^{p,k,s}_t = \zeta^{p,k,s}_u &- \int_u^t \Big[\nabla f^{p,k}(X_r, V^{p,k}_r)\cdot D_u D_s\left(X_r, V^{p,k}_r\right) \\ &+ D^{\perp}_u\left(X_r, V^{p,k}_r\right)\cdot\nabla^2 f^{p,k}(X_r, V^{p,k}_r)\cdot D_s\left(X_r, V^{p,k}_r\right)\Big]dr \\ &+ \int_u^t D_u\zeta^{p,k,s}_r dW_r + \int_u^t D_u\Psi^{p,k,s}_r d\tilde{N}_r, \end{aligned}$$

which is well-defined under (H′), see Lemma 4.3.1-(v). After possibly passing to a suitable version,

$$\zeta^{p,k,s}_t = D_t\Upsilon^{p,k,s}_t = \Upsilon^{p,k,s,t}_t,$$

where $(\Upsilon^{p,k,s,u}, \zeta^{p,k,s,u}, \Psi^{p,k,s,u})$ is the unique solution to FBSDE

$$\begin{cases} \Upsilon^{p,k,s,u}_t = \nabla g^k(X^k_T)\chi^{k,s,u}_T + \chi^{k,u}_T\nabla^2 g^k(X^k_T)\chi^{k,s}_T - \int_t^T \zeta^{p,k,s,u}_r dW_r - \int_t^T \Psi^{p,k,s,u}_r d\tilde{N}_r \\ \qquad + \int_t^T \Big[\nabla f^{p,k}(X^k_r, V^{p,k}_r)\cdot\left(\chi^{k,s,u}_r, \Psi^{p,k,s,u}_r\right) \\ \qquad + \left(\chi^{k,u}_r, \Psi^{p,k,u}_r\right)\cdot\nabla^2 f^{p,k}(X^k_r, V^{p,k}_r)\cdot\left(\chi^{k,s}_r, \Psi^{p,k,s}_r\right)^{\perp}\Big]dr \\ \chi^{k,s,u}_t = \nabla\sigma^k(X^k_{s^-})\chi^{k,u}_s + \int_s^t\left[\nabla b^k(X^k_r)\chi^{k,s,u}_r + \nabla^2 b^k(X^k_r)\chi^{k,u}_r\chi^{k,s}_r\right]dr \\ \qquad + \int_s^t\left[\nabla\sigma^k(X^k_r)\chi^{k,s,u}_r + \nabla^2\sigma^k(X^k_r)\chi^{k,u}_r\chi^{k,s}_r\right]dW_r \\ \qquad + \int_s^t\left[\nabla\gamma^k(X^k_{r^-})\chi^{k,s,u}_{r^-} + \nabla^2\gamma^k(X^k_{r^-})\chi^{k,u}_{r^-}\chi^{k,s}_{r^-}\right]dN_r \end{cases}$$

in which $D_u D_s X^k = D_u\chi^{k,s} = \chi^{k,s,u}$. By Lemma 4.3.1-(v), we can apply exactly the same computation as in Part 1 to $(\chi^{k,s,u}, \Upsilon^{p,k,s,u}, \zeta^{p,k,s,u}, \Psi^{p,k,s,u})$ instead of $(\chi^{k,s}, \Upsilon^{p,k,s}, \zeta^{p,k,s}, \Psi^{p,k,s})$. Let us first prove in a same way that:

$$\left\|\chi^{k,s,u}\right\|^2_{\mathcal{S}^2}$$

is bounded by some constant dependent on (λ, p) to be determined. For any $\xi \in \{b, \sigma, \gamma\}$, denoting $\xi^k_r = \xi^k(X^k_r)$,

$$\begin{aligned} \left|\nabla\xi^k_r\chi^{k,s,u}_r + \nabla^2\xi^k_r\chi^{k,u}_r\chi^{k,s}_r\right| &\leq C\left|\chi^{k,s,u}_r\right| + C\left|\chi^{k,u}_r\chi^{k,s}_r\right| \\ &\leq C\left|\chi^{k,s,u}_r\right| + \frac{C}{2}\left(\left|\chi^{k,u}_r\right|^2 + \left|\chi^{k,s}_r\right|^2\right). \end{aligned}$$

In consequence,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\le r\le t}\left|\chi_r^{k,s,u}\right|^2\right]\\
\le\ & 4\mathbb{E}\left|\nabla\sigma^k(X_{s^-}^k)\chi_s^{k,u}\right|^2+4T\mathbb{E}\int_s^t\left|\nabla b_r^k\chi_r^{k,s,u}+\nabla^2 b_r^k\chi_r^{k,u}\chi_r^{k,s}+\left(\nabla\gamma_r^k\chi_{r^-}^{k,s,u}+\nabla^2\gamma_r^k\chi_{r^-}^{k,u}\chi_{r^-}^{k,s}\right)\lambda\right|^2dr\\
&+4\mathbb{E}\int_s^t\left|\nabla\sigma_r^k\chi_r^{k,s,u}+\nabla^2\sigma_r^k\chi_r^{k,u}\chi_r^{k,s}\right|^2dr+4\mathbb{E}\int_s^t\left|\nabla\gamma_r^k\chi_{r^-}^{k,s,u}+\nabla^2\gamma_r^k\chi_{r^-}^{k,u}\chi_{r^-}^{k,s}\right|^2\lambda dr\\
\le\ & C\mathbb{E}\left|\chi_{s^-}^{k,u}\right|^2+C(1+\lambda+\lambda^2)\int_s^t\left(\left|\chi_r^{k,s,u}\right|^2+\left|\chi_r^{k,u}\right|^4+\left|\chi_r^{k,s}\right|^4\right)dr\\
\le\ & \bar{C}_{x,\lambda}^{2,e}+C(1+\lambda+\lambda^2)\int_s^t\left(\left|\chi_r^{k,u}\right|^4+\left|\chi_r^{k,s}\right|^4\right)dr+C(1+\lambda+\lambda^2)\int_s^t\left|\chi_r^{k,s,u}\right|^2dr\\
\le\ & \bar{C}_{x,\lambda}^{2,e}+\bar{C}_{x,\lambda}^{4,e}+C(1+\lambda+\lambda^2)\int_s^t\mathbb{E}\left[\sup_{s\le t\le r}\left|\chi_t^{k,s,u}\right|^2\right]dr
\end{aligned}$$

by (4.41) and (to be proven just afterwards)

$$\sup_{0\le s\le T}\left\|\chi^{k,s}\right\|_{\mathcal{S}^4_{[s,T]}}^4=\sup_{0\le s\le T}\mathbb{E}\left[\sup_{s\le t\le T}\left|\chi_t^{k,s}\right|^4\right]\le\bar{C}_{x,\lambda}^{4,e}\tag{4.47}$$

where $\bar{C}_{x,\lambda}^{4,e}$ is defined in (4.34). Applying Gronwall's lemma to $t\to\mathbb{E}\left[\sup_{s\le r\le t}\left|\chi_r^{k,s,u}\right|^2\right]$, we get

$$\sup_{0\le s,u\le T}\left\|\chi^{k,s,u}\right\|_{\mathcal{S}^2_{[s,T]}}^2\le\bar{C}_{x,\lambda}^{2,e}+\bar{C}_{x,\lambda}^{4,e}.\tag{4.48}$$

Estimate (4.47) can be proven in the same way as (4.41). By Cauchy-Schwartz's inequality and Burkholder-Davis-Gundy's inequality, we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\le r\le t}\left|\chi_r^{k,s}\right|^4\right]&\le 64\mathbb{E}\left|\sigma^k(X_{s^-}^k)\right|^4+64T^3\mathbb{E}\int_s^t\left|\nabla b^k(X_r^k)\chi_r^{k,s}+\nabla\gamma^k(X_{r^-}^k)\chi_{r^-}^{k,s}\lambda\right|^4dr\\
&\quad+64C\mathbb{E}\int_s^t\left|\nabla\sigma^k(X_r^k)\chi_{r^-}^{k,s}\right|^4dr+64C\lambda^2\mathbb{E}\int_s^t\left|\nabla\gamma^k(X_{r^-}^k)\chi_r^{k,s}\right|^4\lambda dr\\
&\le C\left(1+\mathbb{E}\left|X_{s^-}^k\right|^4\right)+C(1+\lambda^2+\lambda^4)\int_s^t\mathbb{E}\left|\chi_r^{k,s}\right|^4dr.
\end{aligned}$$

Besides, with the same kind of arguments that we used in Lemma 4.2.1, we have

$$\left\|X^k\right\|_{\mathcal{S}^4}^4\le\bar{C}_{x,\lambda}^{4,e}.$$

Indeed,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le s\le t}|X_s^k|^4\right]&\le 64x^4+64T^3\mathbb{E}\left[\int_0^t\left|b^k(X_s^k)+\gamma^k(X_s^k)\lambda\right|^4ds\right]\\
&\quad+64C\mathbb{E}\left[\int_0^t\left|\sigma^k(X_s^k)\right|^4ds\right]+64C\lambda^2\mathbb{E}\left[\int_0^t\left|\gamma^k(X_s^k)\right|^4ds\right]\\
&\le Cx^4+C\left(1+\lambda^2+\lambda^4\right)\int_0^t\left(1+\mathbb{E}|X_s^k|^4\right)ds\\
&\le C\left(x^4+1+\lambda^2+\lambda^4\right)+C\left(1+\lambda^2+\lambda^4\right)\int_0^t\mathbb{E}\left[\sup_{0\le u\le s}|X_u^k|^4\right]ds
\end{aligned}$$

and the result follows from Gronwall's lemma. By again Gronwall's lemma, we get then

$$\mathbb{E}\left[\sup_{s\leq r\leq t}\left|\chi_r^{k,s}\right|^4\right]\leq \bar{C}_{x,\lambda}^{4,e}e^{C(1+\lambda^2+\lambda^4)(t-s)}$$

leading to (4.47). We are now ready to estimate the quantity

$$\sup_{0\leq s,u\leq T}\left\|\Upsilon^{p,k,s,u}\right\|^2_{\mathcal{S}^2_{[s,T]}}.$$

It is easy to compute with the same ideas as in Part 1, if you notice (using shorthand notation $f^{p,k}(X_r^k,V_r^{p,k})=f_r^{p,k}$) that the driver of BSDE (4.47) is such that

$$\begin{aligned}
&\left|\nabla f_r^{p,k}\cdot\left(\chi_r^{k,s,u},\Psi_r^{p,k,s,u}\right)+\left(\chi_r^{k,u},\Psi_r^{p,k,u}\right)\cdot\nabla^2 f_r^{p,k}\cdot\left(\chi_r^{k,s},\Psi_r^{p,k,s}\right)^{\perp}\right|\\
\leq\ &\left|\nabla_x f_r^{p,k}\chi_r^{k,s,u}\right|+\left|\nabla_v f_r^{p,k}\Psi_r^{p,k,s,u}\right|+\left|\chi_r^{k,u}\nabla^2_{xx}f_r^{p,k}\chi_r^{k,s}\right|+\left|\Psi_r^{p,k,u}\nabla^2_{vv}f_r^{p,k}\Psi_r^{p,k,s}\right|\\
&+\left|\chi_r^{k,u}\nabla^2_{xv}f_r^{p,k}\Psi_r^{p,k,s}\right|+\left|\Psi_r^{p,k,u}\nabla^2_{vx}f_r^{p,k}\chi_r^{k,s}\right|\\
\leq\ &C(1+\lambda p)\left|\chi_r^{k,s,u}\right|+C\lambda p\left|\Psi_r^{p,k,s,u}\right|+C(1+\lambda p)k\left|\chi_r^{k,u}\chi_r^{k,s}\right|+C\lambda pk\left|\Psi_r^{p,k,u}\Psi_r^{p,k,s}\right|\\
&+C(1+\lambda p)k\left|\chi_r^{k,u}\Psi_r^{p,k,s}\right|+C\lambda pk\left|\Psi_r^{p,k,u}\chi_r^{k,s}\right|\\
\leq\ &C(1+\lambda p)\left|\chi_r^{k,s,u}\right|+C\lambda p\left|\Psi_r^{p,k,s,u}\right|+C(1+\lambda p)k\left(\left|\chi_r^{k,u}\right|^2+\left|\chi_r^{k,s}\right|^2\right)+C\lambda pk\left|\chi_r^{k,s}\right|^2\\
&+C\lambda pk\left(\left|\Psi_r^{p,k,u}\right|^2+\left|\Psi_r^{p,k,s}\right|^2\right)+C(1+\lambda p)k\left|\Psi_r^{p,k,s}.\right|^2
\end{aligned}$$

In consequence, considering as usual a partition $([s_l,s_{l+1}])_{l=0,\dots,K-1}$ of $[s,T]$ with length of intervals equal to $\delta_K=\frac{T-s}{K}$,

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s,u}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}}+\left\|\zeta^{p,k,s,u}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)}+\left\|\Psi^{p,k,s,u}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}\\
\leq\ &C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^2\\
&+C\delta_K\mathbb{E}\left[\int_{s_l}^{s_{l+1}}\left|\nabla f_r^{p,k}\cdot\left(\chi_r^{k,s,u},\Psi_r^{p,k,s,u}\right)+\left(\chi_r^{k,u},\Psi_r^{p,k,u}\right)\cdot\nabla^2 f_r^{p,k}\cdot\left(\chi_r^{k,s},\Psi_r^{p,k,s}\right)^{\perp}\right|^2dr\right]\\
\leq\ &C\mathbb{E}\left|\Upsilon^{p,k,s,u}_{s_{l+1}}\right|^2+C\delta_K^2(1+\lambda p)^2\left\|\chi^{k,s,u}\right\|^2_{\mathcal{S}^2_{[s,T]}}+C\delta_K\lambda p^2\left\|\Psi^{p,k,s,u}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}\\
&+C\delta_K^2\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2\sup_{0\leq s\leq T}\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}}\\
&+C\delta_K\left[\frac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2\sup_{0\leq s\leq T}\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s,T]}(N)}.
\end{aligned}$$

When again choosing $\delta_K = \frac{1}{2C\lambda p^2}$,

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s,u}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\zeta^{p,k,s,u}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\Psi^{p,k,s,u}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\leq\ & C\mathbb{E}\left|\Upsilon^{p,k,s,u}_{s_{l+1}}\right|^2 + C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\chi^{k,s,u}\right\|^2_{\mathcal{S}^2_{[s,T]}} \\
&+C\frac{1}{(\lambda p^2)^2}\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2 \sup_{0\leq s\leq T}\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}} \\
&+C\frac{1}{\lambda p^2}\left[\frac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2 \sup_{0\leq s\leq T}\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s,T]}(N)}.
\end{aligned}$$

Up to the estimation (to be proven afterwards)

$$\sup_{0\leq s\leq T}\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s,T]}(N)} \leq E^4_x(\lambda,p) \tag{4.49}$$

together with

$$\left|\nabla g^k(X^k_T)\chi^{k,s,u}_T + \chi^{k,u}_T\nabla^2 g^k(X^k_T)\chi^{k,s}_T\right| \leq C|\chi^{k,s,u}_T| + \tfrac{C}{2}\left(|\chi^{k,u}_T|^2+|\chi^{k,s}_T|^2\right)$$

and estimates 4.48 and (4.47), this allows us to conclude by iteration on l that

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s,u}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \\
\leq\ & C^K\left(\sup_{0\leq s\leq T}\mathbb{E}|\chi^{k,s}_T|^4\right) + C^K\left(1+\tfrac{(1+\lambda p)^2}{\lambda p^2}\right)\left\|\chi^{k,s,u}\right\|^2_{\mathcal{S}^2_{[s,T]}} \\
&+C^K\tfrac{1}{\lambda p^2}\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2\sup_{0\leq s\leq T}\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}} \\
&+C^K\left[\tfrac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2\sup_{0\leq s\leq T}\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s,T]}(N)} \\
\leq\ & \left(\bar{C}^{2,e}_{x,\lambda}+\bar{C}^{4,e}_{x,\lambda}\right)e^{C\lambda p^2}\left(1+\tfrac{(1+\lambda p)^2}{\lambda p^2}\right) \\
&+\bar{C}^{4,e}_{x,\lambda}e^{C\lambda p^2}\tfrac{1}{\lambda p^2}\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2 + E^4_x(\lambda,p)e^{C\lambda p^2}\left[\tfrac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2.
\end{aligned}$$

This leads to

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s,u}\right\|^2_{\mathcal{S}^2_{[s,T]}} \leq K\max_{l=0,\dots,K-1}\left\|\Upsilon^{p,k,s,u}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \\
\leq\ & \left(\bar{C}^{2,e}_{x,\lambda}+\bar{C}^{4,e}_{x,\lambda}\right)e^{C\lambda p^2}\left((1+\lambda p)^2+\lambda p^2\right) + \bar{C}^{4,e}_{x,\lambda}e^{C\lambda p^2}\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2 \\
&+E^4_x(\lambda,p)e^{C\lambda p^2}\lambda p^2\left[\frac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2.
\end{aligned} \tag{4.50}$$

It remains us to compute $E_x^4(\lambda,p)$ in (4.49). This can be done by using the same arguments as in Part 1 for $\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,T]}(N)}$. We have on the same partition of $[s,T]$ as usual

$$\begin{aligned}
&\left\|\Upsilon^{p,k,s}\right\|^4_{\mathcal{S}^4_{[s_l,s_{l+1}]}} + \left\|\zeta^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s_l,s_{l+1}]}(W)} + \left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s_l,s_{l+1}]}(N)} \\
\leq\ & C\mathbb{E}\left[\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^4 + \left|\int_{s_l}^{s_{l+1}} \left|\nabla f^{p,k}(X^k_r,V^{p,k}_r)\cdot\left(\chi^{k,s}_r,\Psi^{p,k,s}_r\right)\right| dr\right|^4\right] \\
\leq\ & C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^4 + C\delta_K^3\mathbb{E}\left[\int_{s_l}^{s_{l+1}} \left|\nabla f^{p,k}(X^k_r,V^{p,k}_r)\cdot\left(\chi^{k,s}_r,\Psi^{p,k,s}_r\right)\right|^4 dr\right] \\
\leq\ & C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^4 + C\delta_k^4(1+\lambda p)^4\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}} + C\delta_K^3\lambda^3p^4\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s_l,s_{l+1}]}(N)}.
\end{aligned}$$

Choosing $\delta_K = \left(\frac{1}{2C\lambda^3K_p^4}\right)^{\frac{1}{3}}$, we get

$$\left\|\Upsilon^{p,k,s}\right\|^4_{\mathcal{S}^4_{[s_l,s_{l+1}]}} + \left\|\zeta^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s_l,s_{l+1}]}(N)} \leq C\mathbb{E}\left|\Upsilon^{p,k,s}_{s_{l+1}}\right|^4 + C\frac{(1+\lambda p)^4}{(\lambda^3K_p^4)^{\frac{4}{3}}}\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}}. \tag{4.51}$$

By iterating on s_l until $s_K = T$, with $K \leq T\left(2C\lambda^3K_p^4\right)^{\frac{1}{3}}$ and (4.47), we get for any $l = 0,\dots,K-1$

$$\begin{aligned}
\left\|\Upsilon^{p,k,s}\right\|^4_{\mathcal{S}^4_{[s_l,s_{l+1}]}} &\leq C^{K-k}\mathbb{E}\left|\nabla g^k(X^k_T)\chi^{k,s}_T\right|^4 + (K-k)C^{K-k}\frac{(1+\lambda p)^4}{(\lambda^3K_p^4)^{\frac{4}{3}}}\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}} \\
&\leq C^K\left(1+\frac{(1+\lambda p)^4}{\lambda^3K_p^4}\right)\left\|\chi^{k,s}\right\|^4_{\mathcal{S}^4_{[s,T]}} \\
&\leq \bar{C}^{4,e}_{x,\lambda}e^{C\lambda K_p^{\frac{4}{3}}}\left(1+\frac{(1+\lambda p)^4}{\lambda^3K_p^4}\right).
\end{aligned}$$

Summing up over $l = 0,\dots,K-1$ in equation (4.51) leads finally to

$$\begin{aligned}
\left\|\Psi^{p,k,s}\right\|^4_{\mathrm{L}^4_{[s,T]}(N)} &\leq 2C\sum_{l=0}^{K-1}\mathbb{E}\left|Y^p_{s_{l+1}}\right|^4 + 2CK\frac{(1+\lambda p)^4}{(\lambda^3p^4)^{\frac{4}{3}}}\|X\|^2_{\mathcal{S}^2} \\
&\leq \bar{C}^{4,e}_{x,\lambda}e^{C\lambda K_p^{\frac{4}{3}}}\left(\lambda p^{\frac{4}{3}} + \frac{(1+\lambda p)^4}{(\lambda^3K_p^4)^{\frac{2}{3}}}\right) + \bar{C}^{4,e}_{x,\lambda}\frac{(1+\lambda p)^4}{\lambda^3p^4} \\
&\leq \bar{C}^{4,e}_{x,\lambda}e^{C\lambda K_p^{\frac{4}{3}}}\left(\lambda p^{\frac{4}{3}} + \left(1+\frac{1}{\lambda}\right)\frac{(1+\lambda p)^4}{(\lambda^3K_p^4)^{\frac{2}{3}}}\right) := E^4_x(\lambda,p).
\end{aligned}$$

Injecting this estimate in (4.50), we get

$$\begin{aligned}
\left\|\zeta^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} &\leq \left(\bar{C}^{2,e}_{x,\lambda}+\bar{C}^{4,e}_{x,\lambda}\right)e^{C\lambda p^2}\left((1+\lambda p)^2+\lambda p^2\right)+\bar{C}^{4,e}_{x,\lambda}e^{C\lambda p^2}\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2\\
&+\bar{C}^{4,e}_{x,\lambda}e^{2C\lambda p^2}\lambda p^2\left(\lambda p^{\frac{4}{3}}+\left(1+\frac{1}{\lambda}\right)\frac{(1+\lambda p)^4}{(\lambda^3 K_p^4)^{\frac{2}{3}}}\right)\left[\frac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2\\
&\leq \left(\bar{C}^{2,e}_{x,\lambda}+\bar{C}^{4,e}_{x,\lambda}\right)e^{C\lambda p^2}\left((1+\lambda p)^2+\lambda p^2\right)+\bar{C}^{4,e}_{x,\lambda}e^{C\lambda p^2}\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2\\
&+\bar{C}^{4,e}_{x,\lambda}e^{2C\lambda p^2}\left((\lambda p^2)^2+\frac{1}{\lambda}\left(1+\frac{1}{\lambda}\right)\right)\left[\frac{1}{\lambda}(1+\lambda p)^2+\lambda p^2\right]k^2\\
&\leq \left(\bar{C}^{2,e}_{x,\lambda}+\bar{C}^{4,e}_{x,\lambda}\right)e^{C\lambda p^2}\left((1+\lambda p)^2+\lambda p^2\right)\\
&+\bar{C}^{4,e}_{x,\lambda}e^{C\lambda p^2}\left(1+\frac{1}{\lambda}+\lambda^3 p^4\right)\left[(1+\lambda p)^2+(\lambda p)^2\right]k^2\\
&\leq E^2_x(\lambda,p)+E^3_x(\lambda,p)k^2.
\end{aligned}$$

recalling (4.32) and (4.33). This leads thus to

$$\left\|\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,t]}(W)} \leq \sup_{0\leq s\leq T}\left\|\zeta^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}|t-s| \leq \left(E^2_x(\lambda,p)+E^3_x(\lambda,p)k^2\right)|t-s|.$$

Let us now bound

$$\sup_{0\leq s\leq T}\left\|\Psi^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}$$

by some constant depending on (λ,p) to be determined explicitly. We know that there exists a version of $\Psi^{p,k,s}$ such that

$$\Psi^{p,k,s}_t=\Upsilon^{p,k,s}_t(t,x^{k,1}_t)-\Upsilon^{p,k,s}_t(t,x^{k,2}_t),\quad \forall 0<s\leq t\leq T$$

where $x^{k,1}_t=\chi^{k,s}_{t^-}+\nabla\gamma^k(X^k_{t^-})\chi^{k,s}_{t^-}\in \mathrm{L}^2(\mathcal{F}_t)$, $x^{k,2}_t=\chi^{k,s}_{t^-}\in\mathrm{L}^2(\mathcal{F}_t)$. $\Upsilon^{p,k,s}(t,x)$ denotes the value process of the solution $(\Upsilon^{p,k,s}(t,x),\zeta^{p,k,s}(t,x),\Psi^{p,k,s}(t,x))$ to BSDE (4.39), when the state $\chi^{k,s}$ is replaced by $\chi^{k,s}(t,x)$. We shall use the same kind of reasoning as in Part 1 and shorthand notations

$$\begin{aligned}
&&\delta\chi^{k,s}&=\chi^{k,s}(t,x^{k,1}_t)-\chi^{k,s}(t,x^{k,2}_t)\\
\forall A\in\{\Upsilon,\zeta,\Psi\},\ &&\delta A^{p,k,s}&=A^{p,k,s}(t,x^{k,1}_t)-A^{p,k,s}(t,x^{k,2}_t)\\
\forall \xi\in\{b,\sigma,\gamma,g\},\ &&\delta\left(\nabla\xi^k_r\chi^{k,s}_r\right)&=\nabla\xi^k(X^k_r(t,x^{k,1}_t))\chi^{k,s}_r(t,x^{k,1}_t)-\nabla\xi^k(X^k_r(t,x^{k,2}_t))\chi^{k,s}_r(t,x^{k,2}_t).
\end{aligned}$$

It is easy to show that

$$\left\|\delta\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}}\leq\bar{C}^{2,e}_{x,\lambda}. \tag{4.52}$$

Indeed, $\forall s\leq t\leq u\leq T$,

$$\begin{aligned}
\delta\chi^{k,s}_u &= x^{k,1}_t-x^{k,2}_t+\int_t^u\left[\delta\left(\nabla b^k_r\chi^{k,s}_r\right)+\delta\left(\nabla\gamma^k_r\chi^{k,s}_r\right)\lambda\right]dr\\
&+\int_t^u\delta\left(\nabla\sigma^k_r\chi^{k,s}_r\right)dW_r+\int_t^u\delta\left(\nabla\gamma^k_r\chi^{k,s}_r\right)d\tilde{N}_r.
\end{aligned}$$

Besides, by Lemma 4.3.1-(iv), for any $\xi \in \{b, \sigma, \gamma\}$, we have

$$\begin{aligned}\left|\delta\left(\nabla\xi_r^k\chi_r^{k,s}\right)\right| &\leq \left|\delta\nabla\xi_r^k\cdot\chi_r^{k,s}(t,x_t^{k,1})\right| + \left|\nabla\xi^k(X_r^k(t,x_t^{k,2}))\cdot\delta\chi_r^{k,s}\right| \\ &\leq C\left|\chi_r^{k,s}(t,x_t^{k,1})\right| + C\left|\delta\chi_r^{k,s}\right|\end{aligned}$$

so that by Jensen's and Doob's inequalities,

$$\begin{aligned}\mathbb{E}\left[\sup_{t\leq r\leq u}\left|\delta\chi_r^{k,s}\right|^2\right] &\leq 4\mathbb{E}\left|x_t^{k,1}-x_t^{k,2}\right|^2 + 4T\mathbb{E}\int_t^u\left|\delta\left(\nabla b_r^k\chi_r^{k,s}\right)+\delta\nabla\left(\gamma_r^k\chi_r^{k,s}\right)\lambda\right|^2 dr \\ &+ 4\mathbb{E}\int_t^u\left|\delta\left(\nabla\sigma_r^k\chi_r^{k,s}\right)\right|^2 dr + 4\mathbb{E}\int_t^u\left|\delta\left(\nabla\gamma_r^k\chi_r^{k,s}\right)\right|^2\lambda dr \\ &\leq C\mathbb{E}\left|\chi_{t^-}^{k,s}\right|^2 + C\left(1+\lambda+\lambda^2\right)\int_t^u\left|\chi_r^{k,s}(t,x_t^{k,1})\right|^2 dr \\ &+ C\left(1+\lambda+\lambda^2\right)\int_t^u\mathbb{E}\left|\delta\chi_r^{k,s}\right|^2 dr.\end{aligned}$$

By (4.41), we get

$$\mathbb{E}\left[\sup_{t\leq r\leq u}\left|\delta\chi_r^{k,s}\right|^2\right] \leq \bar{C}_{x,\lambda}^{2,e} + C\left(1+\lambda+\lambda^2\right)\int_t^u\mathbb{E}\left[\sup_{t\leq r\leq l}\left|\delta\chi_r^{k,s}\right|^2\right]dl,$$

so that Gronwall's lemma implies

$$\left\|\delta\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq \bar{C}_{x,\lambda}^{2,e}.$$

In a second time, let us give a bound to

$$\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}}.$$

We have $\forall s\leq t\leq u\leq T$,

$$\delta\Upsilon_u^{p,k,s} = \delta\left(\nabla g_T^k\chi_T^{k,s}\right) + \int_u^T\delta\left(\nabla f_r^{p,k}\cdot(\chi_r^{k,s},\Psi_r^{p,k,s})\right)dr - \int_u^T\delta\zeta_r^{p,k,s}dW_r - \int_u^T\delta\psi_r^{p,k}d\tilde{N}_r.$$

As usual, introduce a partition of $[t,T]$, namely $([s_l,s_{l+1}])_{l=0,\dots,K-1}$ with same length $\delta_K = \frac{T-t}{K}$ and write

$$\begin{aligned}&\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\delta\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq\ & C\mathbb{E}\left|\delta\Upsilon_{s_{l+1}}^{p,k,s}\right|^2 + C\delta_K\mathbb{E}\left[\int_{s_l}^{s_{l+1}}\left|\delta\left(\nabla f_r^{p,k}\cdot(\chi_r^{k,s},\Psi_r^{p,k,s})\right)\right|^2 dr\right].\end{aligned}$$

Lemma 4.3.1 for $f^{p,k}$ gives

$$\begin{aligned}\left|\delta\left(\nabla f_r^{p,k}\cdot(\chi_r^{k,s},\Psi_r^{p,k,s})\right)\right| &\leq \left|\delta\left(\nabla_x f_r^{p,k}\cdot\chi_r^{k,s}\right)\right| + \left|\delta\left(\nabla_v f_r^{p,k}\cdot\Psi_r^{p,k,s}\right)\right| \\ &\leq \left|\delta\nabla_x f_r^{p,k}\cdot\chi_r^{k,s}(t,x_t^{k,1})\right| + \left|\nabla_x f^{p,k}(X_r^k(t,x_t^{k,2}))\cdot\delta\chi_r^{k,s}\right| \\ &+ \left|\delta\nabla_v f_r^{p,k}\cdot\Psi_r^{p,k,s}(t,x_t^{k,1})\right| + \left|\nabla_v f^{p,k}(X_r^k(t,x_t^{k,2}))\cdot\delta\Psi_r^{p,k,s}\right| \\ &\leq C(1+\lambda p)\left|\chi_r^{k,s}(t,x_t^{k,1})\right| + C(1+\lambda p)\left|\delta\chi_r^{k,s}\right| \\ &+ C\lambda p\left|\Psi_r^{p,k,s}(t,x_t^{k,1})\right| + C\lambda p\left|\delta\Psi_r^{p,k,s}\right|\end{aligned}$$

so that

$$\begin{aligned}
&\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\delta\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\leq\ & C\mathbb{E}\left|\delta\Upsilon^{p,k,s}_{s_{l+1}}\right|^2 + C\delta_K^2(1+\lambda p)^2\left\|\delta\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + C\delta_K^2(1+\lambda p)^2\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \\
&+C\delta_K\lambda p^2\left\|\delta\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} + C\delta_K\lambda p^2\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[t,T]}(N)}.
\end{aligned}$$

Choosing $\delta_K = \frac{1}{2C\lambda p^2}$ and iterating on s_l until $s_K = T$, we get by (4.41) and (4.52)

$$\begin{aligned}
&\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\delta\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\leq\ & C\mathbb{E}\left|\delta\Upsilon^{p,k,s}_{s_{l+1}}\right|^2 + C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\delta\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}} + C\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}} + C\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[t,T]}(N)} \\
\leq\ & C^{K-k}\mathbb{E}\left|\delta\left(\nabla g_T^k\chi_T^{k,s}\right)\right|^2 + (K-k)C^{K-k}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left(\left\|\delta X^k\right\|^2_{\mathcal{S}^2_{[t,T]}} + \left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}}\right) \\
&+C^{K-k}\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[t,T]}(N)} \\
\leq\ & C^K\left(1+\frac{(1+\lambda p)^2}{\lambda p^2}\right)\bar{C}^{2,e}_{x,\lambda} + C^K\frac{1}{\lambda p^2}\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[t,T]}(N)}.
\end{aligned}$$

Besides, summing up over $l = 0, \ldots, K-1$ in equation (4.42) of Part 1 leads to

$$\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,T]}(N)} \leq \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2 + \left(1+\frac{1}{\lambda}\right)(1+\lambda p)^2\right)$$

which implies

$$\begin{aligned}
&\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\delta\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\
\leq\ & \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(1+\left(1+\frac{1}{\lambda}\right)\frac{(1+\lambda p)^2}{\lambda p^2}\right)
\end{aligned}$$

and thus

$$\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq K\max_{l=0,\ldots,K-1}\left\|\delta\Upsilon^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} \leq \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2 + \left(1+\frac{1}{\lambda}\right)(1+\lambda p)^2\right).$$

In consequence,

$$\left\|\Psi^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}} \leq \bar{C}^{2,e}_{x,\lambda}e^{C\lambda p^2}\left(\lambda p^2 + \left(1+\frac{1}{\lambda}\right)(1+\lambda p)^2\right),$$

leading to

$$\begin{aligned}
\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,t]}(N)} &\leq \left\|\Psi^{p,k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}\lambda|t-s| \\
&\leq \bar{C}^{2,e}_{x,\lambda}(1+\lambda)e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|t-s| \leq E^2_x(\lambda,p)|t-s|.
\end{aligned}$$

Injecting the estimates for $\left\|\zeta^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,t]}(W)}$ and $\left\|\Psi^{p,k,s}\right\|^2_{\mathrm{L}^2_{[s,t]}(N)}$ in (4.47) allow us to conclude that

$$z_1 \leq \left(E_x^2(\lambda,p) + E_x^3(\lambda,p)k^2\right)|t-s|.$$

Step 2. To estimate the second part z_2, we use the same kind of arguments as in Part 2. Denoting

$$\forall A \in \{\Upsilon,\zeta,\Psi\}, \quad \begin{array}{l}\delta A^{p,k} = A^{p,k,t} - A^{p,k,s} \\ \delta\chi^k = \chi^{k,t} - \chi^{k,s}\end{array}$$

we have $\forall s \leq t \leq u \leq T$,

$$\delta\Upsilon_u^{p,k} = \nabla g_T^k \delta\chi_T^k + \int_u^T \nabla f_r^{p,k} \cdot \left(\delta\chi_r^k, \delta\Psi_r^{p,k}\right) dr - \int_u^T \delta\zeta_r^{p,k} dW_r - \int_u^T \delta\Psi_r^{p,k} d\tilde{N}_r.$$

As usual, introduce a partition of $[t,T]$ in $K \geq 1$ sub-intervals $([s_l, s_{l+1}])_{l=0,\ldots,K-1}$ with same length $\delta_K = \frac{T-t}{K} \leq \frac{T}{K}$. By Lemma 4.3.1-(v),

$$\left|\nabla f_r^{p,k} \cdot \left(\delta\chi_r^k, \delta\Psi_r^{p,k}\right)\right| \leq C(1+\lambda p)|\delta\chi_r^k| + C\lambda p|\delta\Psi_r^{p,k}|$$

so that

$$\begin{aligned} &\left\|\delta\Upsilon^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta\zeta^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \left\|\delta\Psi^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq\ & C\mathbb{E}\left|\delta\Upsilon_{s_{l+1}}^{p,k}\right|^2 + C\delta_K^2(1+\lambda p)^2\left\|\delta\chi^k\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + C\delta_K\lambda p^2\left\|\delta\Psi^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)}. \end{aligned}$$

Choosing $\delta_K = \frac{1}{2C\lambda p^2}$, iterating on s_l until $s_K = T$, together with the estimate (4.45), we obtain

$$\begin{aligned} &\left\|\delta\Upsilon^{p,k}\right\|^2_{\mathcal{S}^2_{[s_l,s_{l+1}]}} + \left\|\delta\zeta^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(W)} + \frac{1}{2}\left\|\delta\Psi^{p,k}\right\|^2_{\mathrm{L}^2_{[s_l,s_{l+1}]}(N)} \\ \leq\ & C^{K-k}\mathbb{E}\left|\nabla g_T^k \delta\chi_T^k\right|^2 + (K-k)C^{K-k}\frac{(1+\lambda p)^2}{(\lambda p^2)^2}\left\|\delta\chi^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \\ \leq\ & C^K\left(1 + \frac{(1+\lambda p)^2}{\lambda p^2}\right)\left\|\delta\chi^k\right\|^2_{\mathcal{S}^2_{[t,T]}}. \end{aligned}$$

But (this will be shown just afterwards)

$$\left\|\delta\chi^k\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq \bar{C}^{2,e}_{x,\lambda}(1+\lambda)^2|t-s| \tag{4.53}$$

so that

$$\left\|\delta\Upsilon^{p,k}\right\|^2_{\mathcal{S}^2_{[t,T]}} \leq \bar{C}^{2,e}_{x,\lambda}(1+\lambda)^2 e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|t-s|$$

and

$$z_2 \leq \bar{C}^{2,e}_{x,\lambda}(1+\lambda)^2 e^{C\lambda p^2}\left(\lambda p^2 + (1+\lambda p)^2\right)|t-s| \leq E_x^2(\lambda,p)|t-s|.$$

Let us show (4.53). By definition, we have for any $s \leq t \leq u \leq T$,

$$\begin{aligned} \delta\chi_u^k = {}& \sigma^k(X_{t^-}^k) - \sigma^k(X_{s^-}^k) + \int_s^t \left(\nabla b_r^k \chi_r^{k,s} dr + \nabla\sigma_r^k \chi_r^{k,s} dW_r + \nabla\gamma_r^k \chi_r^{k,s} dN_r\right) \\ &+ \int_t^u \left(\nabla b_r^k \delta\chi_r^k dr + \nabla\sigma_r^k \delta\chi_r^k dW_r + \nabla\gamma_r^k \delta\chi_r^k dN_r\right) \end{aligned}$$

so that (together with estimates (4.46) of Part 2 and (4.41) of Part 1)

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le r\le u}\left|\delta\chi_r^k\right|^2\right] &\le C\left(|t-s|+\mathbb{E}\left|X_{t^-}^k-X_{s^-}^k\right|^2\right)+C(1+\lambda+\lambda^2)\int_s^t\mathbb{E}\left|\chi_r^{k,s}\right|^2dr\\
&\quad+C(1+\lambda+\lambda^2)\int_t^u\mathbb{E}\left|\delta\chi_r^k\right|^2dr\\
&\le \bar{C}_{x,\lambda}^{2,e}|t-s|+C(1+\lambda+\lambda^2)\left\|\chi^{k,s}\right\|^2_{\mathcal{S}^2_{[s,T]}}|t-s|\\
&\quad+C(1+\lambda+\lambda^2)\int_t^u\mathbb{E}\left[\sup_{t\le s\le r}\left|\delta\chi_s^k\right|^2\right]dr\\
&\le \bar{C}_{x,\lambda}^{2,e}(1+\lambda)^2|t-s|+C(1+\lambda+\lambda^2)\int_t^u\mathbb{E}\left[\sup_{t\le s\le r}\left|\delta\chi_s^k\right|^2\right]dr.
\end{aligned}$$

By Gronwall's lemma, it implies $\mathbb{E}\left[\sup_{t\le r\le T}\left|\delta\chi_r^k\right|^2\right]\le\bar{C}_{x,\lambda}^{2,e}(1+\lambda)^2|t-s|$. Adding z_1 to z_2 allows to conclude that

$$\max_{n=0,\dots,N-1}\sup_{t_n\le t\le t_{n+1}}\mathbb{E}\left|Z_t^{p,k}-Z_{t_n}^{p,k}\right|^2\le\left(E_x^2(\lambda,p)+E_x^3(\lambda,p)k^2\right)|\pi|.$$

Chapter 5

Numerical applications

In this chapter, we provide numerical results[1] obtained by using the method introduced in Chapter 4 for solving two impulse control problems. We compare our results to available benchmarks. The fully implementable algorithms used in both practical cases are precisely described. In particular, the conditional expectations involved in the backward procedures are approximated by a least squares Monte Carlo approach. The use of BSDEs with jumps is difficult in practice. To our knowledge, only Elie [48] has published numerical experiments involving BSDEs with jumps (without constaint). In our framework, the main difficulty comes from the adjustment of approximation parameters λ (jump intensity) et p (penalization coefficient), recall the global convergence rate of our approximate method in Theorem 4.4.1.

In Section 5.1, we deal with a stochastic control problem of forest management proposed by Øksendal and Sulem [94]. This problem belongs to the well-known class of stochastic rotation problems (see as well Willassen [113]) and is for us a good practical example since the approximation of the exact solution as the solution to a penalized BSDE with jumps can be justified rigorously with respect to any assumptions which have been introduced in previous chapters. An explicit solution to this problem is also available and this allows us to illustrate the convergence of our approximate method. Accurate results are obtained and the method is efficient on this particular case.

In a second time, see Section 5.2, our method is applied for the valuation of simple Swing options. This constitutes a multiple optimal stopping time problem which can be reformulated as a particularly degenerated three-dimensional impulse control problem. The method is clearly less competitive but we have been able to obtain converged valuation results, for a small number of exercises rights $n_{\max} \leq 2$.

In Section 5.3, we propose a fully implementable algorithm for the valuation of gas storage facilities. This problem has been presented in Part I and boils down to an optimal switching problem, see (1.6). Its solution can be represented as the solution to a BSDE with constrained jumps, recall (1.24), so that the same methodology as the one introduced in chapters 3 and 4 is applicable. By using our penalization procedure, we propose a purely simulation-based method which allows in particular to handle with the *degenerate, controlled and constrained* inventory level variable. As explained below, to obtain relevant valuation results, the intensity measure λ has to be adapted to the *a priori* optimal behavior of the storage. The algorithm need to be modified in this sense to be efficient and getting valuation results with such a numerical method is a challenging problem, which is left for further research.

[1] All the numerical algorithms developped in the framework of this thesis have been implemented in `C++`.

5.1 A problem of optimal forest management

5.1.1 Problem formulation

Let us consider a problem of optimal forest management, in which the dynamics of the forest biomass is given by

$$dX_t^u = bdt + \sigma dW_t,$$

where $b > 0$, $\sigma > 0$ and W is a standard Brownian motion. An impulse strategy $u = (\tau_k)_{k\geq 1}$ is an increasing sequence of times at which we decide to cut down the forest and replant it, namely

$$X_{\tau_k}^u = 0, \quad \forall k \geq 1$$

with a cost

$$c + \theta X_{\tau_k^-}^u,$$

where $c > 0$ is a fixed cost and $0 \leq \theta < 1$ a proportional cost per unit of cut biomass. We set by convention $\tau_0 = 0$. The objective is to find the optimal strategy which maximizes the expected total discounted net profit, that is

$$v(t,x) = \sup_{u=(\tau_k)_{k\geq 1}} \mathbb{E}\left[\sum_{k\geq 1} e^{-\rho(t+\tau_k)}\left(X_{\tau_k^-}^u - c - \theta X_{\tau_k^-}^u\right) | X_t^u = x\right] \tag{5.1}$$

where $\rho > 0$ is a given discounting coefficient. In the following, the intervention gain will be denoted by

$$\kappa(x) := (1-\theta)x - c.$$

Exact solution in infinite horizon

v defined in (5.1) is solution to the QVI

$$\min\left\{-\partial_t v(t,x) + \rho v(t,x) - \mathcal{L}v(t,x); v(t,x) - \mathcal{H}v(t,x)\right\} = 0, \quad \forall (t,x) \in [0,+\infty) \times \mathbb{R} \tag{5.2}$$

in which

$$\begin{aligned}\mathcal{L}v(t,x) &= bD_x v(t,x) + \frac{1}{2}\sigma^2 D_x^2 v(t,x),\\ \mathcal{H}v(t,x) &= v(t,0) + \kappa(x).\end{aligned}$$

An exact solution is easily available (see in [94]):

$$v(t,x) = e^{-\rho t} v(x), \quad \forall (t,x) \in [0,T] \times [0,+\infty), \tag{5.3}$$

where

$$v(x) = \begin{cases} \frac{1-\theta}{r} e^{-r(x^*-x)} & \text{if } x < x^* \\ \frac{1-\theta}{r} e^{-rx^*} + (1-\theta)x - c & \text{else} \end{cases}$$

with x^*, the unique solution on $(0,+\infty)$ to

$$e^{-rx^*} + rx^* - 1 - \frac{r}{1-\theta}c = 0 \tag{5.4}$$

and

$$r = \frac{1}{\sigma^2}\left(\sqrt{b^2 + 2\rho\sigma^2} - b\right) > 0.$$

Namely, the optimal strategy consists in cutting the forest every time the biomass reaches the threshold x^*.

The quasi-analytical solution (5.3) will allow us to perform a convergence study of our approximate method based on the resolution of a penalized BSDE with jumps.

5.1.2 Solving the problem using BSDEs with jumps

Associated BSDEs with jumps

Previously-introduced solving approach using a BSDE with constrained jumps is applicable in *finite horizon*. Let T be a finite maturity. To solve the infinite horizon problem (5.1), we will use a relevant terminal condition for the BSDE, namely the (known) exact solution at time T.

Let us introduce a Poisson process N with intensity $\lambda > 0$ and consider the uncontrolled state variable X with dynamics

$$dX_t = bdt + \sigma dW_t - X_{t^-} dN_t, \tag{5.5}$$

If the jump times of N are denoted by $(\mathrm{T}_k)_{k\geq 1}$, we have

$$\begin{cases} dX_t = bdt + \sigma dW_t, \text{ on } [\mathrm{T}_k, \mathrm{T}_{k+1}), \forall k \geq 1, \\ X_{\mathrm{T}_k} = 0, \ \forall k \geq 1. \end{cases}$$

The BSDE with constrained jumps associated to infinite horizon problem (5.1) is

$$\begin{cases} Y_t = v(T, X_T) - \int_t^T \rho Y_s ds - \int_t^T Z_s dW_s - \int_t^T (U_s - \kappa(X_{s^-}))dN_s + \int_t^T dK_s, \forall 0 \leq t \leq T \\ -U_t \geq 0, \forall 0 \leq t \leq T \end{cases} \tag{5.6}$$

in which v is defined in (5.3). Required assumptions of Corollary 2.2.1 are satisfied. Indeed:

- (H_X): the underlying coefficients are such that b and σ are constants and $\gamma(x) := -x$.
- (H_Y): the BSDE coefficients are $g(x) := v(T, x)$, $f(y) := -\rho y$ and $\kappa(x) = (1-\theta)x - c$.
- The exact solution v satisfies a linear growth condition, see (5.3).
- It is clear that assumptions (H_1), (H'_1) and (H_2) hold since by construction $v(t, X_t) = Y_t$ and v in (5.3) satisfies a linear growth condition and belongs to $\mathcal{C}^{1,2}$.

As a consequence, the solution to (5.1) is given by the minimal solution to (5.6) when $(X_s)_{t\leq s\leq T} = (X_s^{t,x})_{t\leq s\leq T}$ is the solution to (5.5) starting in x at time t. Let $p > 0$ be a penalization coefficient, the penalized BSDE with jumps associated to (5.6) is

$$Y_t^p = v(T, X_T) - \int_t^T \rho Y_s^p ds - \int_t^T Z_s^p dW_s - \int_t^T (U_s^p - \kappa(X_{s^-}))dN_s + p\int_t^T (U_s^p)^+ \lambda ds. \tag{5.7}$$

Additional assumption (H^*) needed for Theorem 3.2.1 is satisfied as well, see propositions 5.1.1 and 5.1.2.

Proposition 5.1.1. *Assumption* (H*) *holds for impulse control problem* (5.1). *More specifically, if* $u^* = (\tau_k^*)_{k\geq 1}$ *denotes the optimal impulse strategy to infinite horizon problem* (5.1), *then*

$$\mathbb{P}\left\{H^{*,\epsilon}\right\} := \mathbb{P}\left\{\min_{k\geq 0}\left|\tau_{k+1}^* - \tau_k^*\right| \leq \epsilon\right\} = \mathcal{O}_{\epsilon\to 0}\left(\sqrt{\epsilon}\right).$$

Proof. Let $\epsilon > 0$. By definition of the optimal strategy u^* as the sequence of hitting times of the intervention region, we have

$$\forall k \geq 0, \quad \tau_{k+1}^* = \inf\left\{t > \tau_k^* : v(t, X_{t^-}^{u^*}) = v(t, X_{t^-}^{u^*} + \gamma(X_{t^-}^{u^*})) + \kappa(X_{t^-}^{u^*})\right\}.$$

This implies

$$\begin{aligned}
&\mathbb{P}\left\{\min_{k\geq 0}\left|\tau_{k+1}^* - \tau_k^*\right| \leq \epsilon\right\} = \mathbb{P}\left\{\exists k \geq 0, \tau_{k+1}^* \leq \tau_k^* + \epsilon\right\} \\
= \quad &\mathbb{P}\left[\exists k \geq 0, \arg\inf_{t>\tau_k^*}\left\{v(t, X_{t^-}^{u^*}) = v(t, X_{t^-}^{u^*} + \gamma(X_{t^-}^{u^*})) + \kappa(X_{t^-}^{u^*})\right\} \leq \tau_k^* + \epsilon\right]
\end{aligned}$$

We know *a posteriori* that v is increasing in x, cf. (5.3). In consequence

$$\begin{aligned}
\mathbb{P}\left\{\min_{k\geq 0}\left|\tau_{k+1}^* - \tau_k^*\right| \leq \epsilon\right\} &\leq \mathbb{P}\left[\exists k \geq 0, \arg\inf_{t>\tau_k^*}\left\{\kappa(X_{t^-}^{u^*}) \geq 0\right\} \leq \tau_k^* + \epsilon\right] \\
&= \mathbb{P}\left[\exists k \geq 0, \arg\inf_{t>\tau_k^*}\left\{X_{t^-}^{u^*} \geq \frac{c}{1-\theta}\right\} \leq \tau_k^* + \epsilon\right] \\
&\leq \mathbb{P}\left[\exists k \geq 0, \sup_{\tau_k^* \leq t < \tau_k^* + \epsilon} X_t^{u^*} \geq \frac{c}{1-\theta}\right] \\
&= \mathbb{E}\left[\mathbb{P}\left[\exists k \geq 0, \sup_{\tau_k^* \leq t < \tau_k^* + \epsilon}\left\{b(t-\tau_k^*) + \sigma(W_t - W_{\tau_k^*})\right\} \geq \frac{c}{1-\theta}\middle|\tau_k^*\right]\right] \\
&= \mathbb{P}\left[\sup_{0\leq s<\epsilon}\left\{bs + \sigma B_s\right\} \geq \frac{c}{1-\theta}\right]
\end{aligned}$$

where $(B_s)_{s\geq 0} = (W_{s+\tau_k^*} - W_{\tau_k^*})_{s\geq 0}$ is a Brownian motion independent of $\mathcal{F}_{\tau_k^*}$, since $X_{\tau_k^*}^{u^*} = 0$. We recognize the first hitting time of a barrier $c/(1-\theta) > 0$ by a Brownian motion with drift starting at 0. Set $a = c/\left[\sigma(1-\theta)\right]$ and denote by $\mathcal{N}$, the cumulative distribution function of the Normal distribution, then

$$\begin{aligned}
\mathbb{P}\left\{\min_{k\geq 0}\left|\tau_{k+1}^* - \tau_k^*\right| \leq \epsilon\right\} &\leq e^{2ba}\mathcal{N}\left(-\frac{a+b\epsilon}{\sqrt{\epsilon}}\right) + \mathcal{N}\left(-\frac{a-b\epsilon}{\sqrt{\epsilon}}\right) \\
&\sim (1+e^{2ba})\mathcal{N}\left(-\frac{a}{\sqrt{\epsilon}}\right) \\
&\sim \frac{(1+e^{2ba})}{a\sqrt{2\pi}} e^{-\frac{a^2}{\epsilon}}\sqrt{\epsilon} \\
&= \mathcal{O}\left(\sqrt{\epsilon}\right)
\end{aligned}$$

as $\epsilon \to 0$ by $\mathcal{N}(-x) = 1 - \mathcal{N}(x) \sim_{x\to+\infty} \frac{1}{\sqrt{2\pi}}\frac{e^{-x^2}}{x}$. □

Proposition 5.1.2. *If $n^*_{(0,T]}$ denotes the number of impulses in the optimal strategy $u^* = (\tau^*_k)_{k\geq 1}$, then*

$$\mathbb{P}\left(n^*_{(0,T]} > n\right) = \mathcal{O}_{n\to+\infty}\left(e^{-\frac{a^2}{T}n}\frac{1}{\sqrt{n}}\right)$$

in which $a = c/\left[\sigma(1-\theta)\right]$.

Proof. Let $n \in \mathbb{N}^*$. Obviously $\forall k \geq 0, \left|\tau^*_{k+1} - \tau^*_k\right| > \frac{T}{n}$ implies $n^*_{(0,T]} < n$. By the same arguments as in the proof of Proposition 5.1.2, we get

$$\begin{aligned}
\mathbb{P}\left(n^*_{(0,T]} \geq n\right) &\leq \mathbb{P}\left\{\min_{k\geq 0}\left|\tau^*_{k+1} - \tau^*_k\right| \leq \frac{T}{n}\right\} \\
&\leq e^{2ba}\mathcal{N}\left(-\frac{a+b\frac{T}{n}}{\sqrt{\frac{T}{n}}}\right) + \mathcal{N}\left(-\frac{a-b\frac{T}{n}}{\sqrt{\frac{T}{n}}}\right) \\
&\sim (1+e^{2ba})\mathcal{N}\left(-a\sqrt{\frac{n}{T}}\right) \\
&\sim \frac{(1+e^{2ba})\sqrt{T}}{a\sqrt{2\pi}}e^{-\frac{a^2}{T}n}\frac{1}{\sqrt{n}}
\end{aligned}$$

as $n \to +\infty$. □

5.1.3 Discrete-time algorithm using Monte Carlo techniques

On a regular time grid $\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}$, the computation of the discrete-time version X^π of X is straightforward (see for example in Cont and Tankov [37]):

1. Computation of the jump dates $(\mathrm{T}_k)_{k\geq 1}$ of the Poisson process N with intensity λ on $[0,T]$

   (i) $\forall i \geq 1, \delta_i \sim \exp(\lambda)$ i.i.d. while $\sum_{i=1}^k \delta_i \leq T$,
   in which $\exp(\lambda)$ is the exponential distribution with parameter λ .

   (ii) $\forall k \geq 1, \mathrm{T}_k = \sum_{i=1}^k \delta_i$ while $\sum_{i=1}^k \delta_i \leq T$.

2. Computation of X^π

$$\begin{cases}
X^\pi_{t_0} = x \\
\forall t_{n+1} \in \pi : \\
\qquad \text{if } N \text{ does not jump on } (t_n, t_{n+1}] : \\
\qquad\quad X^\pi_{t_{n+1}} = X^\pi_{t_n} + b\Delta t_{n+1} + \sigma\Delta W_{t_{n+1}} \\
\qquad \text{else :} \\
\qquad\quad \mathrm{T}_{k_n} := \text{ last jump time of } N \text{ on } (t_n, t_{n+1}] \\
\qquad\quad X^\pi_{t_{n+1}} = b(t_{n+1} - \mathrm{T}_{k_n}) + \sigma(W_{t_{n+1}} - W_{\mathrm{T}_{k_n}})
\end{cases}$$

   in which $\Delta t_{n+1} := t_{n+1} - t_n$ and $\Delta W_{t_{n+1}} := W_{t_{n+1}} - W_{t_n}$.

3. As a byproduct, we get also the compensated Poisson increment $\Delta\tilde{N}_{t_{n+1}}$

$$\forall t_{n+1} \in \pi, \quad \Delta\tilde{N}_{t_{n+1}} = \sharp\left\{k \geq 1, t_n < \mathrm{T}_k \leq t_{n+1}\right\} - \lambda\Delta t_{n+1}$$

The discrete-time approximation of solution (Y^p, Z^p, U^p) to (5.7) is denoted by

$$(Y^{p,\pi}, Z^{p,\pi}, U^{p,\pi}).$$

As the driver of the penalized BSDE (5.7) does not depend on Z^p, it is sufficient to compute $(Y^{p,\pi}, U^{p,\pi})$ on π. Besides, an explicit scheme is available, since the BSDE driver is linear in Y^p. The backward scheme is thus:

$$\begin{cases} Y_{t_N}^{p,\pi} = v(t_N, X_{t_N}^\pi) \\ \forall t_n \in \pi, t_n < t_N : \\ \qquad U_{t_n}^{p,\pi} = \frac{1}{\lambda \Delta t_n} \mathbb{E}_{t_n}\left[Y_{t_{n+1}}^{p,\pi} \Delta \tilde{N}_{t_{n+1}} \right] + \kappa(X_{t_n}^\pi) \\ \qquad Y_{t_n}^{p,\pi} = \frac{1}{1+\rho \Delta t_{n+1}} \left(\mathbb{E}_{t_n}\left[Y_{t_{n+1}}^{p,\pi} \right] + \left[p \left(U_{t_n}^{p,\pi} \right)^+ - U_{t_n}^{p,\pi} + \kappa(X_{t_n}^\pi) \right] \lambda \Delta t_{n+1} \right) \end{cases} \tag{5.8}$$

in which $\mathbb{E}_{t_n}[\cdot] := \mathbb{E}\left[\cdot | X_{t_n}^\pi \right]$. For a sufficiently large penalization coefficient $p > 0$ and enough small time step $|\pi|$, the value $Y_0^{p,\pi}$ gives an approximation to solution $v(0,x)$ in (5.3).

Monte Carlo technique for computing conditional expectations estimators

Estimators of the conditional expectations $\mathbb{E}_{t_n}$ are computed by a classical least squares Monte Carlo technique. It consists in simulating $M \geq 1$ paths of X^π

$$X^{\pi,(m)}, \forall m = 1, \dots, M$$

and compute processes $(Y^{p,\pi}, U^{p,\pi})$ backward in time on each Monte Carlo sample

$$(Y^{p,\pi,(m)}, U^{p,\pi,(m)}), \forall m = 1, \dots, M$$

by the backward procedure

1. Initialization: $Y_{t_N}^{p,\pi,(m)} = v(t_N, X_{t_N}^{\pi,(m)})$, $\forall m \leq M$.
2. Backward induction: for $n = N-1, \dots, 0$, $\forall m \leq M$,

$$\begin{cases} U_{t_n}^{p,\pi,(m)} = \frac{1}{\lambda \Delta t_n} \mathbb{E}^M \left[Y_{t_{n+1}}^{p,\pi} \Delta \tilde{N}_{t_{n+1}} | X_{t_n}^\pi = X_{t_n}^{\pi,(m)} \right] + \kappa(X_{t_n}^{\pi,(m)}) \\ Y_{t_n}^{p,\pi,(m)} = \frac{1}{1+\rho \Delta t_{n+1}} \Big(\mathbb{E}^M \left[Y_{t_{n+1}}^{p,\pi} | X_{t_n}^\pi = X_{t_n}^{\pi,(m)} \right] \\ \qquad\qquad + \left[p \left(U_{t_n}^{p,\pi,(m)} \right)^+ - U_{t_n}^{p,\pi,(m)} + \kappa(X_{t_n}^{\pi,(m)}) \right] \lambda \Delta t_{n+1} \Big) \end{cases}$$

in which conditional expectations estimators are denoted by $\mathbb{E}^M$.
For $G_{t_{n+1}}^\pi \in \left\{ Y_{t_{n+1}}^{p,\pi}, Y_{t_{n+1}}^{p,\pi} \Delta \tilde{N}_{t_{n+1}} \right\}$, the estimation

$$\varepsilon_{t_n}^{(m)} := \mathbb{E}^M \left[G_{t_{n+1}}^\pi | X_{t_n}^\pi = X_{t_n}^{\pi,(m)} \right]$$

of the exact conditional expectation

$$\mathbb{E}\left[G_{t_{n+1}}^\pi | X_{t_n}^\pi = X_{t_n}^{\pi,(m)} \right]$$

is computed by least squares regression of $\left(G^{\pi,(m)}_{t_{n+1}}\right)_{m\leq M}$ on $\left(\psi_1(X^{\pi,(m)}_{t_n}),\dots,\psi_b(X^{\pi,(m)}_{t_n})\right)_{m\leq M}$ where $(\psi_l)_{l=1,\dots,b}$ are local basis functions (see the precise definition of these functions in Bouchard and Warin [21]). Namely

$$\varepsilon^{(m)}_{t_n} = \sum_{l=1}^{b} \beta_l \psi_l(X^{\pi,(m)}_{t_n}),$$

where $\beta = (\beta_1,\dots,\beta_b)$ minimizes the least squares Monte Carlo error, that is

$$\beta = \arg\min_{\alpha\in\mathbb{R}^b} \sum_{m=1}^{M} \left| G^{\pi,(m)}_{t_{n+1}} - \sum_{l=1}^{b} \alpha_l \psi_l(X^{\pi,(m)}_{t_n}) \right|.$$

5.1.4 Numerical results

The numerical experiments have been performed on two sets of parameters with a finite maturity $T = 3$.

First set of parameters

Fixed cut cost	$c = 0.5$
Cut cost	$\theta = 0.5$
Discounting factor	$\rho = 1$
Drift	$b = 4$
Volatility	$\sigma = 1$

Preliminary observations We use the continuous time quasi-analytical solution in (5.3) as a benchmark. In particular, the optimal forest cut threshold solution to (5.4) is such that

$$x^* \approx 3.247.$$

Figure 5.1 shows the mean optimal number of forest cuts and the expected optimal gain as function of x at time 0, when using a discrete-time computation with $N = 1600$ time steps. These values are computed by forward Monte Carlo simulation with 1 million paths, knowing the optimal cut threshold x^*, see also Figure 5.2.

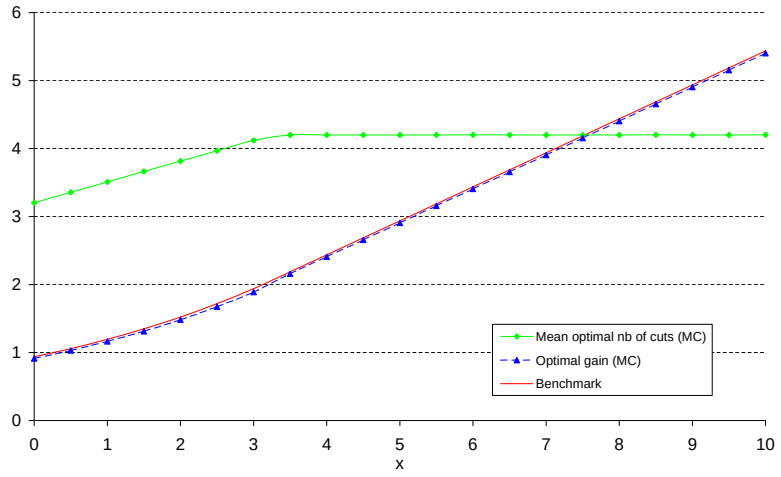

Figure 5.1: Quasi-analytical solution in infinite horizon.

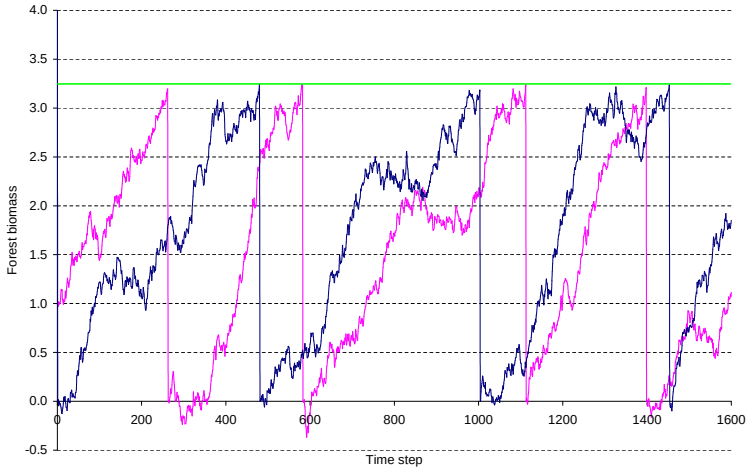

Figure 5.2: Simulated trajectories of the optimal forest biomass.

The mean optimal number of forest cuts $\mathbb{E}\left[n^*_{(0,T]}\right]$ vary from 3.20 to 4.20 (see Figure 5.1) and we report in Figure 5.3 the distribution of $n^*_{(0,T]}$ for different initial biomass values x (distribution computed with 65500 samples).

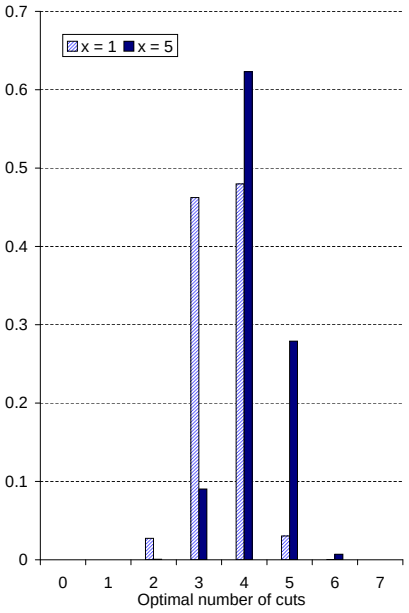

Figure 5.3: Distribution of the number of optimal cuts.

Results obtained when solving the penalized BSDE with jumps Let us now present the valuation results given by the numerical algorithm described in Paragraph 5.1.3. We use $N = 1600$ time steps, $M = 1$ million Monte Carlo paths and $b = 100$ basis functions (we numerically observed that the number of basis functions required to obtain converged values is

particularly high). We report in Table 5.1 the approximate optimal gain for different values of p and λ as $x = 5$ at time 0 (the benchmark gives 2.937). The gain values are means over 5 valuations and the relative standard deviation is written in brackets.

	$p = 1.5$		$p = 2$	
λ	Optimal gain	Relative error to the benchmark	Optimal gain	Relative error to the benchmark
1	2.079 (0.018 %)	29.23%	2.277 (0.015 %)	22.46%
2	2.493 (0.049 %)	15.12%	2.626 (0.043 %)	10.61%
3	2.656 (0.068 %)	9.59%	2.756 (0.057 %)	6.17%
4	2.739 (0.029 %)	6.76%	2.820 (0.032 %)	3.98%
5	2.789 (0.029 %)	5.03%	2.859 (0.020 %)	2.65%
6	2.826 (0.096 %)	3.78%	2.888 (0.101 %)	1.69%
7	2.846 (0.098 %)	3.12%	2.901 (0.095 %)	1.25%
8	2.865 (0.162 %)	2.46%	2.916 (0.161 %)	0.74%
9	2.879 (0.189 %)	1.99%	2.927 (0.197 %)	0.35%

Table 5.1: Approximation of the solution at $x = 5$, when varying λ.

We retrieve the expected monotone convergence in the penalization parameter p. And as shown in Figure 5.4, for fixed p, we observe a monotone convergence in λ for reasonable values of λ. A simple least squares estimation of the approximation error by a power function gives a behavior in $\mathcal{O}(n^{-1.06})$, see Figure 5.5. When increasing much more p, we need to refine sharply the discrete-time grid to preserve the convergence. As expected, the numerical method seems also to be much more sensitive to p than to λ.

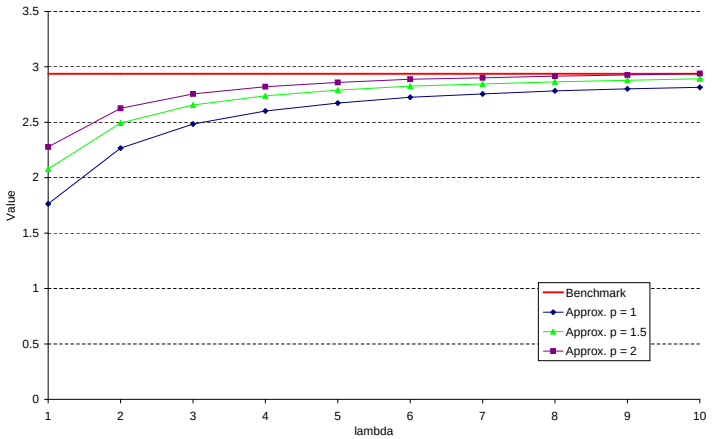

Figure 5.4: Approximate optimal gain as function of λ at $x = 5$.

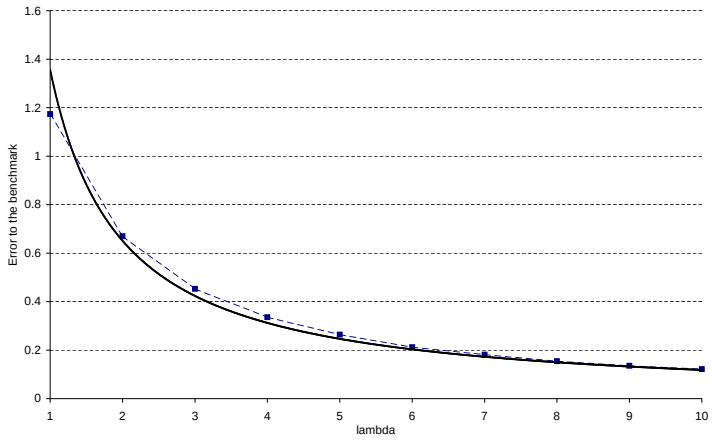

Figure 5.5: Error between the approximate optimal gain and the benchmark as function of λ for $p = 1$ at $x = 5$.

We report finally in Figure 5.6 the approximate optimal gain as function of x at time 0 when $p = 1.5$ for different values of λ. Recall that the mean number of jumps of the Poisson process N over the period $[0, T = 3]$ is equal to 3λ. As shown in Figure 5.6 and as expected, the solution is under-estimated in the intervention region ($x \geq x^*$) for too small values of λ. Indeed, the value process does not jump often enough and the algorithm has difficulties to capture the whole set of possible cutting times.

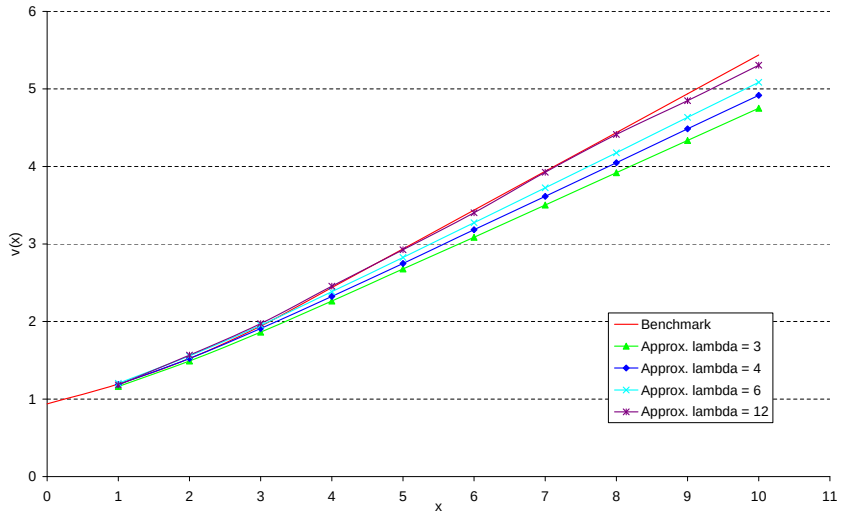

Figure 5.6: Optimal gain as function of the initial forest biomass.

Second set of parameters

Fixed cut cost	$c = 0.5$
Cut cost	$\theta = 0.8$
Discounting factor	$\rho = 0.5$
Drift	$b = 4$
Volatility	$\sigma = 1$

Unless specified otherwise, we use the same solving parameters as in previous Paragraph.

Preliminary observations In this case the optimal cut threshold solution to (5.4) is $x^* \approx 7.327$. We present in Figure 5.7 the mean values of $n^*_{(0,T]}$ and $v(0,x)$ computed in discrete time by Monte Carlo simulation. The distribution of $n^*_{(0,T]}$ is reported in Figure 5.8. With this set of parameter (in particular, a bigger cut cost), the number of optimal cuts is on average smaller than in previous example, and at the same time less dispersed.

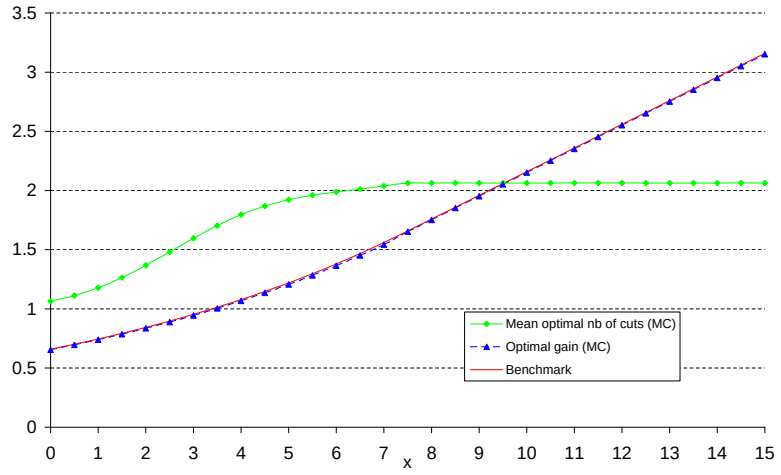

Figure 5.7: Quasi-analytical solution in infinite horizon.

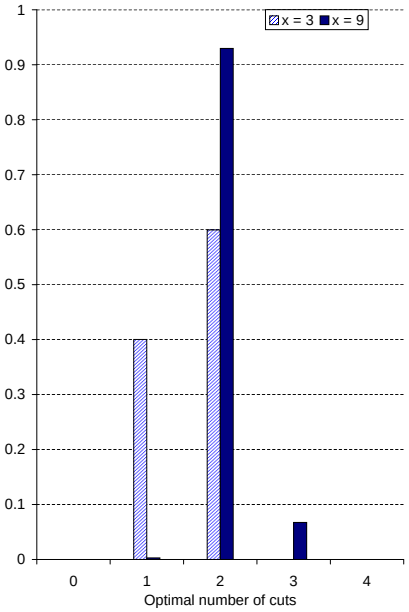

Figure 5.8: Distribution of the number of optimal cuts.

Results obtained when solving the penalized BSDE with jumps We report in Figure 5.9 the approximate optimal gain at $x = 10$ when increasing reasonably p (the computed values are mean over 5 samples and the relative standard deviation is less than 0.75%).

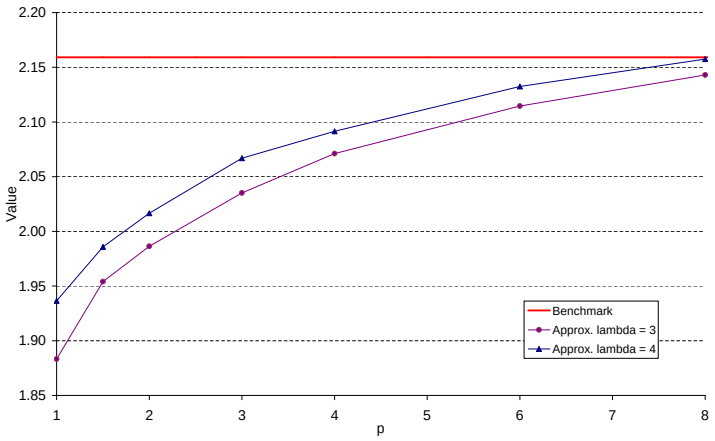

Figure 5.9: Convergence in p of the approximate optimal gain at $x = 10$.

For fixed $p = 4$, the behavior of the approximate optimal gain at $x = 4$ when varying λ is reported in Figure 5.10, when using a fixed number of time steps $N = 1600$ and a fixed number of Monte Carlo paths $M = 1$ million. See also Table 5.2: the computed values are mean over 5 samples (the relative standard deviation is written in brackets) and the benchmark gives 1.079.

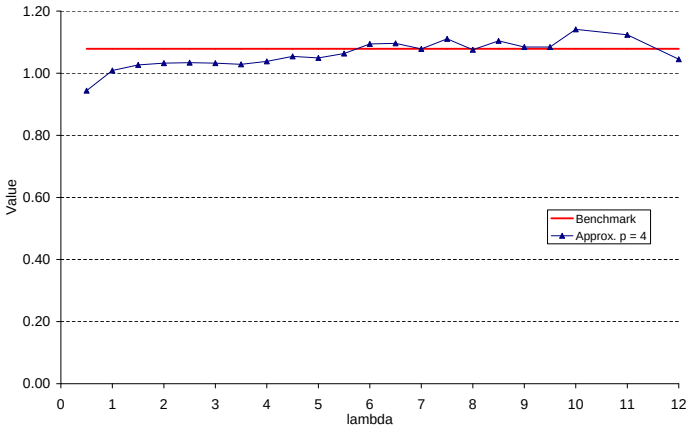

Figure 5.10: Behavior in λ of the approximate optimal gain at $x = 4$.

λ	Optimal gain	Relative error to the benchmark
0.5	0.944 (0.019 %)	12.51 %
1	1.009 (0.082 %)	6.48 %
2	1.032 (0.053 %)	4.29 %
3	1.032 (0.315 %)	4.28 %
4	1.038 (0.608 %)	3.76 %
5	1.049 (0.994 %)	2.73 %
6	1.094 (1.769 %)	1.44 %
7	1.078 (1.238 %)	0.03 %
8	1.076 (1.634 %)	0.28 %
9	1.084 (2.731 %)	0.54 %
10	1.141 (5.321 %)	5.81 %
12	1.045 (6.890 %)	3.12 %

Table 5.2: Approximation of the solution at $x = 4$, when varying λ.

As already mentioned, in our numerical method, λ needs to be sufficiently large to obtain converged values. However, increasing too much λ leads to numerical instabilities: namely, the variance of the method grows. This variance growth is due to the fact that if the penalized solution Y^p jumps more often (increasing λ), the contraint penalizing the driver is more often activated (U^p positive).

Figure 5.11 shows finally the approximate optimal gain as function of x when $p = 4$ (computed values are mean over 5 samples).

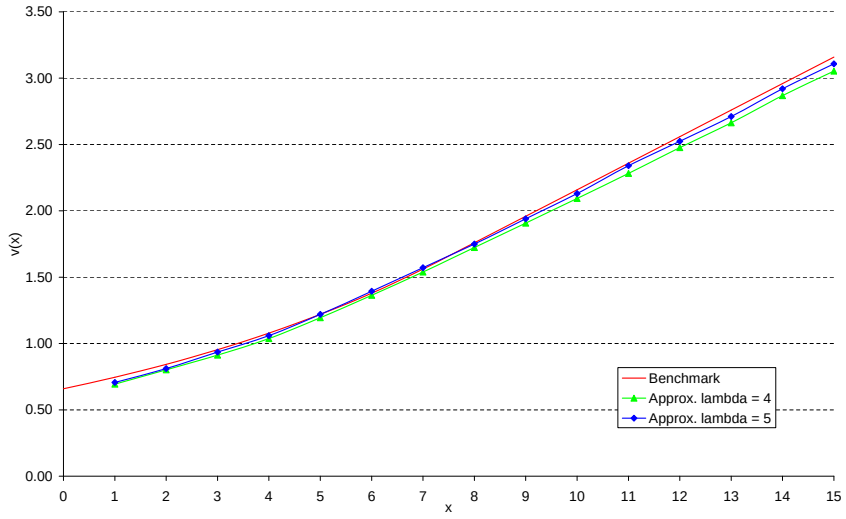

Figure 5.11: Optimal gain as function of the initial forest biomass.

On this case of study, we have been able to observe a monotone convergence in λ of our approximate valuation method. This holds for (small) fixed penalization parameter p and a sufficiently fine time step $|\pi|$. This constitutes a new result which is consistent with the approximation error obtained in Theorem 4.4.1. However, the Monte Carlo algorithm introduced is sensitive to the jump intensity λ and all the more to the penalization parameter p. When increasing these parameters, the variance of the approximate method explodes. This would require the use of a finer time grid and much more Monte Carlo samples.

5.2 Swing options valuation

5.2.1 Problem formulation

Let us consider a classical Swing option, in which the holder of the option is given a maximal number of exercise rights, say $n_{\max} \geq 1$, and has the opportunity to sell or buy whenever he wants over a time period $[0, T]$ an underlying asset against a fixed strike price. We shall denote by ϕ the reward function corresponding to the profit made at each exercise date and by P the underlying asset spot price. We concentrate here on the risk-neutral Black and Scholes framework in which $r > 0$ is a constant interest rate and the spot price process is defined by

$$P_t = P_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma W_t}, \quad \forall t \geq 0 \tag{5.9}$$

where $\sigma > 0$ denotes the volatility coefficient.

A delay $\delta > 0$ between two consecutive exercise dates is introduced. Indeed, without any delay, the optimal strategy would consists in $n_{\max}$ simultaneous exercise at an unique optimal date (so that this option is equivalent to $n_{\max}$ identical American options). The value of such an option can be written as the solution to the following multiple optimal stopping time problem

$$v^{(n_{\max})}(t,p) = \sup_{u=(\tau_k)_{k\geq 1} \in \mathcal{U}^{\delta}_{(0,T]}} \mathbb{E}\left[\sum_{k\geq 1} e^{-r\tau_k}\phi(P^{t,p}_{\tau_k})\right] \tag{5.10}$$

in which a strategy $u = (\tau_k)_{k\geq 1}$ is said to be admissible and belongs to $\mathcal{U}^{\delta}_{(t,T]}$ iff it is an increasing sequence of $\mathbb{F}^W$-stopping time valued in $(t,T]$ (set by convention $\tau_t = 0$) which satisfied the constraint on delay, that is

$$\forall k \geq 1, \tau_{k+1} - \tau_k \geq \delta.$$

As a multiple optimal stopping time problem, this problem can be formulated as a particular impulse control problem. An impulse control corresponds to a sequence of exercise date and the intervention gain is written as the payoff function ϕ multiplied by an indicator function which allows to satisfy both the constraint on the number of exercise rights and the constraint on delay between exercise dates. Namely

$$v(t,p,0,0) = \sup_{u=(\tau_k)_{k\geq 1}} \mathbb{E}\left[\sum_{\substack{k\geq 1 \\ t<\tau_k\leq T}} e^{-r\tau_k}\phi(P_{\tau_k}) \mathbb{1}_{\left\{ \left(\Theta^{k-1}_{\tau_k^-}\geq\delta\right)\cap\left(Q^u_{\tau_k^-}<n_{\max}\right)\right\}} \Big| P_t = p, Q^u_t = \Theta^u_t = 0 \right] \tag{5.11}$$

where $u = (\tau_k)_{k\geq 1}$ is the sequence of exercise dates which are $\mathbb{F}^W$-stopping times valued in $(t,T]$ with convention $\tau_t = 0$ and two additional state variables which both are controlled and discontinuous (càdlàg) processes are introduced:

- Q^u counts the number of exercise rights used before considered time

$$Q^u_0 = 0, \quad Q^u_t = \sharp\left\{k \geq 1, \tau_k \leq t\right\}, \forall t \geq 0.$$

- $\Theta^u_t := \Theta^k_t = \inf\left\{t - \tau_k, \tau_k \leq t\right\}$ corresponds to the delay between t and last exercise date

$$\Theta^k_t = t - \tau_k, \forall \tau_k \leq t < \tau_{k+1}, \quad \Theta^k_{\tau_{k+1}} = 0, \quad \forall k \geq 0,$$

  where by convention $\Theta^u_0 = \Theta^0_0 = 0$.

It is straightforward that problem (5.10) is equivalent to impulse control problem (5.11), that is

$$\forall (t,p) \in [0,T]\times\mathbb{R}, \quad v^{(n_{\max})}(t,p) = v(t,p,0,0). \tag{5.12}$$

This implies in particular that assumptions (H^n) and (H^*) introduced in Section (3.2) hold: the impulse control problem (5.11) can be restricted to admissible strategies which admits an almost surely bounded by $n_{\max}$ number of impulses and such that the distance between two consecutive impulse dates is greater than δ.

Impulse control problem (5.11) constitues a strongly degenerate problem. Indeed:

- The state variable is degenerate. In particular, the bidimensional controlled variable is such that

$$\begin{pmatrix} Q^u_t \\ \Theta^u_t \end{pmatrix} = \begin{pmatrix} 0 \\ t \end{pmatrix} + \sum_{\substack{k\geq 1 \\ \tau_k\leq t}} \begin{pmatrix} 1 \\ -\Theta^u_{\tau_k^-} \end{pmatrix}, \forall t \geq 0.$$

- The intervention gain defined by

$$\kappa(p,q,\theta) := \phi(p)\mathbb{1}_{\{(\theta\geq\delta)\cap(q\leq n_{\max}-1)\}}, \quad \forall (p,q,\theta) \in \mathbb{R}\times\mathbb{N}\times\mathbb{R}^+ \tag{5.13}$$

  is discontinuous in p and θ.

Remark 5.2.1. We could use a classical regularization argument to smoothen κ. Indeed, the indicator function in (5.13) might be replaced by some map $I^k \in \mathcal{C}_b^1$ such that

$$I^k \longrightarrow_{k\to+\infty} \mathbb{1}_{\{(\theta\geq\delta)\cap(q\leq n_{\max}-1)\}}.$$

However, we chose to leave the problem as written in (5.11) so as not to add additional (regularization) parameters.

Following lower and upper bounds for the Swing option value are well-known:

$$\sum_{i=0}^{n_{\max}-1} v_{\mathrm{EU}}^{T-\delta i}(t,p) \leq v^{(n_{\max})}(t,p) \leq \left(n_{\max} \wedge \left\lfloor \frac{T}{\delta} \right\rfloor\right) v_{\mathrm{US}}(t,p), \quad \forall (t,p) \in [0,T]\times\mathbb{R}.$$

where

- v_{US} is the price of the American option with payoff ϕ,
- v_{EU}^{θ} is the price of the European option with payoff ϕ and maturity θ.

This ensures in particular the linear growth of the Swing option value $v^{(n_{\max})}(t,p)$ with respect to p.

5.2.2 Swing options valuation by using iteration

The classical method to value such a Swing option, recall formulation (5.10), is based on an iteration on the number of exercise rights, see for example Carmona and Touzi [27]. The Bellman optimality principle (dynamic programming) provides a direct link between the solution $v^{(j)}$ to the same problem as (5.10) but with at maximum $j \leq n_{\max}$ exercise rights and the solution $v^{(j-1)}$ with at maximum $(j-1)$ exercise rights. Namely:

$$\forall j \in \{1,\dots,n_{\max}\}, \forall (t,p) \in [0,T]\times\mathbb{R}, \quad v^{(j)}(t,p) = \sup_{\tau\geq t} \mathbb{E}\left[e^{-r\tau}\Phi^{(j)}(\tau, P_\tau^{t,p})\right],$$

where $\Phi^{(j)}$ corresponds to the benefit of an exercise if it remains j exercise rights, that is

$$\Phi^{(j)}(t,p) = \begin{cases} \phi(p) + \mathbb{E}\left[e^{-r\delta}v^{(j-1)}(t+\delta, P_{t+\delta}^{t,p})\right] & \text{(exercise at time } t \leq T-\delta\text{)}, \\ \phi(p) & \text{(no more exercise right after } t \text{ if } T-\delta < t \leq T\text{)}. \end{cases}$$

The numerical methods issued from this approach by iteration are known to be memory demanding, since to compute the solution at each iteration j you need to keep in memory the values of the solution on the whole state domain from previous step $(j-1)$. For small $n_{\max}$, this approach is however competitive and will be used as a benchmark to our method.

In the following, we describe briefly the discrete-time algorithm based on this iterative approach. Let $N \in \mathbb{N}^*$ and a regular discrete time grid

$$\pi = \{t_0 = 0, t_1, \dots, t_N = T\}$$

such that the time step $\Delta t_{n+1} = t_{n+1} - t_n = |\pi|, \forall n = 0,\dots,N-1$ is a divider of the delay δ. This means in particular that $\forall t_n \in \pi, t_n + \delta \in \pi$.

Let P^π the discrete-time version of the price process P, which is exactly computed by (5.9). The value of the option Swing with 0 exercise right is obviously zero

$$v^{(0)} = 0.$$

Then, the sequence of values of Swing options with j exercise rights $v^{(j)}$, $j = 1, \dots, n_{\max}$ is computed backward in time on π as

$$\begin{cases} v^{(j)}(t_N, p) = \phi(p) \\ \forall t_n \in \pi, T - \delta < t_n < T : \\ \quad v^{(j)}(t_n, p) = \max\left\{\phi(p) \ ; \ e^{-r\Delta t_{n+1}} \mathbb{E}^{t_n,p}\left[v^{(j)}(t_{n+1}, P^\pi_{t_{n+1}})\right]\right\} \\ \forall t_n \in \pi, t_n \leq T - \delta : \\ \quad v^{(j)}(t_n, p) = \max\left\{\phi(p) + e^{-r\delta}\mathbb{E}^{t_n,p}\left[v^{(j-1)}(t_n + \delta, P^\pi_{t_n+\delta})\right] \ ; \ e^{-r\Delta t_{n+1}} \mathbb{E}^{t_n,p}\left[v^{(j)}(t_{n+1}, P^\pi_{t_{n+1}})\right]\right\} \end{cases}$$

in which $\mathbb{E}^{t_n,p}[\cdot] := \mathbb{E}\left[\cdot | P^\pi_{t_n} = p\right]$. Conditional expectations will be estimated by the same least squares Monte Carlo method as the one described in Paragraph 5.1.3.

5.2.3 Solving the problem using BSDEs with jumps

Let us introduce a Poisson process N with intensity $\lambda > 0$ and the uncontrolled state variable (Q, Θ) defined by

$$\begin{aligned} \forall t \geq 0, \quad Q_t &= N_t, \\ \Theta_t &= t - \int_0^t \Theta_{s^-} dN_s. \end{aligned}$$

Let $p > 0$ be a penalization coefficient. The penalized BSDE with jumps associated to problem (5.11) is

$$\begin{aligned} Y^p_t = \kappa(P_T, Q_{T^-}, \Theta_{T^-}) &- \int_t^T r Y^p_s ds - \int_t^T Z^p_s dW_s \\ &- \int_t^T \left(U^p_s - \kappa(P_s, Q_{s^-}, \Theta_{s^-})\right) dN_s + p \int_t^T (U^p_s)^+ \lambda ds, \quad \forall 0 \leq t \leq T. \end{aligned} \tag{5.14}$$

The forward coefficients of the three-dimensional state variable (P, Q, Θ) are strongly degenerate:

$$b(p,q,\theta) := \begin{pmatrix} rp \\ 0 \\ 1 \end{pmatrix}, \ \Sigma(p,q,\theta) := \begin{pmatrix} \sigma p & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ \gamma(p,q,\theta) := \begin{pmatrix} 0 \\ 1 \\ -\theta \end{pmatrix}, \quad \forall (p,q,\theta) \in \mathbb{R} \times \mathbb{N} \times \mathbb{R}^+,$$

but satisfy required assumptions (H_X). On the contrary, the BSDEs coefficients are too irregular, see (5.14) and (5.13), and does not fulfill the required conditions (H_Y). As already mentioned, see Remark 5.2.1, one could smoothen κ. And yet, for numerical purpose, we preferred to keep unchanged the penalized BSDE (5.14) and apply the same methodology as the general one introduced in Chapter 4.

5.2.4 Discrete-time algorithm using Monte Carlo techniques

Let us consider the same discrete-time framework as in Section 5.2.2. Variables (Q, Θ) are exactly computed on the time grid π:

1. Computation of the jump dates $(\mathrm{T}_k)_{k\geq 1}$ of the Poisson process N with jump intensity λ on $[0,T]$.
   The Poisson increments are denoted by
$$\Delta N_{t_{n+1}} = \sharp\left\{k \geq 1, t_n < \mathrm{T}_k \leq t_{n+1}\right\}, \quad \forall t_{n+1} \in \pi.$$
2. Computation of Q^π
$$Q^\pi_{t_0} = 0, \quad Q^\pi_{t_n} = N_{t_n}, \forall t_n \in \pi.$$
3. Computation of Θ^π
$$\begin{cases} \Theta^\pi_{t_0} = 0 \\ \forall t_{n+1} \in \pi : \\ \qquad \text{if } N \text{ does not jump on } (t_n, t_{n+1}] : \\ \qquad\quad \Theta^\pi_{t_{n+1}} = \Theta^\pi_{t_n} + \Delta t_{n+1} \\ \qquad \text{else :} \\ \qquad\quad \mathrm{T}_{k_n} := \text{ last jump time of } N \text{ on } (t_n, t_{n+1}] \\ \qquad\quad \Theta^\pi_{t_{n+1}} = (t_{n+1} - \mathrm{T}_{k_n}) \end{cases}$$

The discrete-time approximation of (Y^p, Z^p, U^p) defined in (5.14) is denoted by
$$\left(Y^{p,\pi}, Z^{p,\pi}, U^{p,\pi}\right).$$

As the driver of the penalized BSDE (5.14) does not depend on Z^p, it is sufficient to compute backward $(Y^{p,\pi}, U^{p,\pi})$ on π. We consider thus the following backward scheme

$$\begin{cases} Y^{p,\pi}_{t_N} = \kappa(P^\pi_{t_N}, Q^\pi_{t_N}, \Theta^\pi_{t_N}) \\ \forall t_n \in \pi, t_n < T : \\ \qquad U^{p,\pi}_{t_n} = \frac{1}{\lambda \Delta t_{n+1}} \mathbb{E}_{t_n}\left[Y^{p,\pi}_{t_{n+1}} \Delta \tilde{N}_{t_{n+1}}\right] + \kappa(P^\pi_{t_n}, Q^\pi_{t_n}, \Theta^\pi_{t_n}) \\ \qquad Y^{p,\pi}_{t_n} = \frac{1}{1+r\Delta t_{n+1}} \left(\mathbb{E}_{t_n}\left[Y^{p,\pi}_{t_{n+1}}\right] + \left[p(U^{p,\pi}_{t_n})^+ - \left(U^{p,\pi}_{t_n} - \kappa(P^\pi_{t_n}, Q^\pi_{t_n}, \Theta^\pi_{t_n})\right)\right] \lambda \Delta t_{n+1}\right) \end{cases} \tag{5.15}$$

in which $\mathbb{E}_{t_n}[\cdot] := \mathbb{E}\left[\cdot | (P^\pi_{t_n}, Q^\pi_{t_n}, \Theta^\pi_{t_n})\right]$ and $\Delta \tilde{N}_{t_{n+1}} = \Delta N_{t_{n+1}} - \lambda \Delta t_{n+1}$.

Monte Carlo based resolution

We use the same least squares Monte Carlo procedure for estimating conditional expectations estimators as the one described in Paragraph 5.1.3. As it is only relevant for real-valued variables (the regression basis functions have compact support), it cannot handle with the (integer-valued) variable Q^π. To overcome this difficulty, at each time step $t_n < T$, the set of M Monte Carlo samples is separated in $n_{\max}+1$ sub-sets corresponding to the samples on which $Q^\pi_{t_n} = 0, 1, \ldots, n_{\max}-1$ and $Q^\pi_{t_n} \geq n_{\max}$. Then, we just need to estimate $n_{\max}$ conditional expectation operators, namely
$$\mathbb{E}\left[\cdot | (P^\pi_{t_n}, Q^\pi_{t_n} = q, \Theta^\pi_{t_n})\right], \forall q \in \{0, 1, \ldots, n_{\max} - 1\}$$

by using the corresponding Monte Carlo samples, since $(Y^{p,\pi}_{t_n}, U^{p,\pi}_{t_n}) = (0,0)$ when $Q^\pi_{t_n} \geq n_{\max}$. Indeed,

$$\begin{aligned} & \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} \Delta\tilde{N}_{t_{n+1}} | (P^\pi_{t_n}, Q^\pi_{t_n} \geq n_{\max}, \Theta^\pi_{t_n})\right] \\ = \; & \mathbb{E}\Big[\underbrace{\mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} | Q^\pi_{t_n} \geq n_{\max}\right]}_{=0} \Delta\tilde{N}_{t_{n+1}} | (P^\pi_{t_n}, Q^\pi_{t_n} \geq n_{\max}, \Theta^\pi_{t_n})\Big] \; = \; 0, \end{aligned}$$

by definition of κ in (5.13) and (5.15) and in the same way

$$\mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} | (P^\pi_{t_n}, Q^\pi_{t_n} \geq n_{\max}, \Theta^\pi_{t_n})\right] \; = \; 0$$

which implies $Y^{p,\pi}_{t_n} = U^{p,\pi}_{t_n} = 0$.

The Monte Carlo based resolution procedure is then the following:

I. Simulation of M i.i.d. paths of $(P^\pi, Q^\pi, \Theta^\pi)$

$$\left(P^{\pi,(m)}, Q^{\pi,(m)}, \Theta^{\pi,(m)}\right), \forall m = 1, \ldots, M.$$

II. Initialization:

$$Y^{p,\pi,(m)}_{t_N} = \phi(P^{\pi,(m)}_{t_N}) \mathbb{1}_{\left\{\left(\Theta^{\pi,(m)}_{t_N} \geq \delta\right) \cap \left(Q^{\pi,(m)}_{t_N} \leq n_{\max}-1\right)\right\}}, \forall m = 1, \ldots, M.$$

III. Computation backward in time of $(U^{p,\pi,(m)}, Y^{p,\pi,(m)})$ on each sample m.
For $n = N-1, \ldots, 0$, set

$$\begin{aligned} \mathcal{M}^q_{t_n} &:= \left\{m = 1, \ldots, M : Q^{\pi,(m)}_{t_n} = q\right\}, \quad \forall q \leq n_{\max} - 1, \\ \mathcal{M}^{n_{\max}}_{t_n} &:= \left\{m = 1, \ldots, M : Q^{\pi,(m)}_{t_n} \geq n_{\max}\right\}. \end{aligned}$$

Then:

1. For any $m \in \mathcal{M}^{n_{\max}}_{t_n}$,
$$U^{p,\pi,(m)}_{t_n} = Y^{p,\pi,(m)}_{t_n} = 0.$$
2. Set $q := n_{\max} - 1$.
3. If $q \geq 0$, for any $m \in \mathcal{M}^q_{t_n}$, the conditional expectations estimators
$$\begin{aligned} \varepsilon^{U,q,(m)}_{t_n} &\approx \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}} \Delta\tilde{N}_{t_{n+1}}\Big|_{\mathcal{M}^q_{t_n}} \mid \left(P^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n}\right)\right] \\ \varepsilon^{Y,q,(m)}_{t_n} &\approx \mathbb{E}\left[Y^{p,\pi}_{t_{n+1}}\Big|_{\mathcal{M}^q_{t_n}} \mid \left(P^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n}\right)\right] \end{aligned}$$
are respectively approximated by least squares regression of $\left(Y^{p,\pi,(m)}_{t_{n+1}} \Delta\tilde{N}^{(m)}_{t_{n+1}}\right)_{m \in \mathcal{M}^q_{t_n}}$ and $\left(Y^{p,\pi,(m)}_{t_{n+1}}\right)_{m \in \mathcal{M}^q_{t_n}}$ on
$$\left(\psi_1(P^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n}), \ldots, \psi_b(P^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n})\right)_{m \in \mathcal{M}^q_{t_n}}$$

with $b := b^q_{t_n}$ basis functions. Then,

$$\begin{cases} U^{p,\pi,(m)}_{t_n} = \frac{1}{\lambda\Delta t_{n+1}}\varepsilon^{U,q,(m)}_{t_n} + \phi(P^{\pi,(m)}_{t_n})\mathbb{1}_{\left\{\Theta^{\pi,(m)}_{t_n}\geq\delta\right\}} \\ Y^{p,\pi,(m)}_{t_n} = \frac{1}{1+r\Delta t_{n+1}}\left(\varepsilon^{Y,q,(m)}_{t_n}\right. \\ \qquad\qquad \left. + \left[p(U^{p,\pi,(m)}_{t_n})^+ - \left(U^{p,\pi,(m)}_{t_n} - \phi(P^{\pi,(m)}_{t_n})\mathbb{1}_{\left\{\Theta^{\pi,(m)}_{t_n}\geq\delta\right\}}\right)\right]\lambda\Delta t_{n+1}\right). \end{cases}$$

4. $q := q-1$ and go to 3.

IV. In particular at time t_0 ($\mathcal{M}^0_{t_0} = \{1,\dots,M\}$ and $\mathcal{M}^q_{t_0} = \emptyset, \forall q \geq 1$) the Swing option price estimator is given by $Y^{p,\pi}_{t_0}$ such that

$$\begin{cases} U^{p,\pi}_{t_0} = \frac{1}{\lambda\Delta t_1}\frac{1}{M}\sum_{m=1}^M \left(Y^{p,\pi,(m)}_{t_1}\Delta\tilde{N}^{(m)}_{t_1}\right) \\ Y^{p,\pi}_{t_0} = \frac{1}{1+r\Delta t_1}\left(\frac{1}{M}\sum_{m=1}^M Y^{p,\pi,(m)}_{t_1} + \left[p(U^{p,\pi}_{t_0})^+ - U^{p,\pi}_{t_0}\right]\lambda\Delta t_1\right). \end{cases}$$

Remark 5.2.2. Let us highlight some features of the above-presented Monte Carlo procedure. At each backward induction date $t_n < T$, we have to estimate in worst cases $2\times n_{\max}$ conditional expectations, performed on each subset $\mathcal{M}^q_{t_n}, q \leq n_{\max}-1$. When $n_{\max}$ increases, much more Monte Carlo samples are needed as each least squares regression requires a sufficient number of samples.

In addition, the number of local basis functions $b^q_{t_n}$ has to be adapted to the number of Monte Carlo samples used for the least squares regression, namely $\mathrm{card}(\mathcal{M}^q_{t_n})$. Thus, we introduce a dynamic choice for $b^q_{t_n}$: it is fixed proportionally to $\mathrm{card}(\mathcal{M}^q_{t_n})$ for any $q \leq n_{\max}-1$ and $t_n < T$.

Finally, a classical technique of data storage in binary files is used to avoid memory overrun[2]. The basic idea is the following: for each $n = 1,\dots,N$, the values for

$$\left(P^{\pi,(m)}_{t_n}, Q^{\pi,(m)}_{t_n}, \Theta^{\pi,(m)}_{t_n}\right), \forall m = 1,\dots,M$$

are sequentially written in a binary file `fic[n].bin` during the forward simulation step I. Then, at each step $n = N,\dots,1$ of the backward resolution (points II. and III.), corresponding file `fic[n].bin` is opened for reading the samples values needed for the computation of $(U^{p,\pi,(m)}, Y^{p,\pi,(m)})$. This technique allows to roughly reduce the memory size required for the computation. However, the cost in computational time is considerable, since file writing/reading operations take much more time than simple matrices handling.

5.2.5 Pricing results

We consider put options with maturity $T = 1$ year, payoff $\phi(p) = (K-p)^+$ and a strike price $K = 100$. The Black and Scholes parameters, see (5.9), are $r = 0.05$, $\sigma = 0.3$ and $s = 100$.

Special case of American options: $n_{\max} = 1$

In the single-exercise case, the additional variable Θ disappears (there is no delay constraint). This helps simplify the Monte Carlo procedure described in Paragraph 5.2.4. In particular, the algorithm implies only two sequences of samples subsets, whether one jump of N occurs before

[2] I am grateful to Xavier Warin for helpful suggestions for improvement in the solving algorithms.

T on the considered path or not: that is $((\mathcal{M}^1)_{t_n})_{n\leq N-1}$ and $((\mathcal{M}^0)_{t_n})_{n\leq N-1}$.

In our numerical experiments, we find out that increasing too much λ makes the variance of the Monte Carlo procedure explode. It would be necessary to increase the number of Monte Carlo samples, which leads to prohibitive computational times (each pricing result presented below was obtained after a computation between 6 and 8 hours). For the same reason (exploding behavior of the penalized BSDE driver), we restrict our numerical experiments to penalization parameters ≤ 5.

The benchmark price for the American put option is 9.88 (by a binomial approach or classical Monte Carlo). We report in Table 5.3 the price given by our method when varying λ and the number of time steps N for a penalization parameter equal to 5. We used 20 million of Monte Carlo paths.

λ \ N	20	40	80	160	320
3	9.89	9.92	9.95	9.94	9.83
4	9.92	9.96	9.99	9.97	9.83
5	9.95	9.99	10.02	9.98	9.76

Table 5.3: Approximate prices of an American option with $p = 5$.

In all the experiments that we performed in this simple case, we numerically observed that the limiting prices of our method (with respect to N) are below the benchmark value: this is due to penalization.

Swing Options with $n_{\max} = 2$

We consider time delays $\delta = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}$.

The benchmark prices for the Swing put option with 2 exercise rights are 19.27, 18.77 and 18.21 respectively (computed with the method described in Paragraph 5.2.2, $N = 200$ time steps and $M = 5$ million of Monte Carlo paths). We report in Figures 5.12, 5.13 and 5.14 the corresponding approximate prices when varying λ and N for a penalization parameter equal to 5 and 10 (we used 40 million of Monte Carlo paths and $N = 20, 40, 80, 160, 320, 640$).

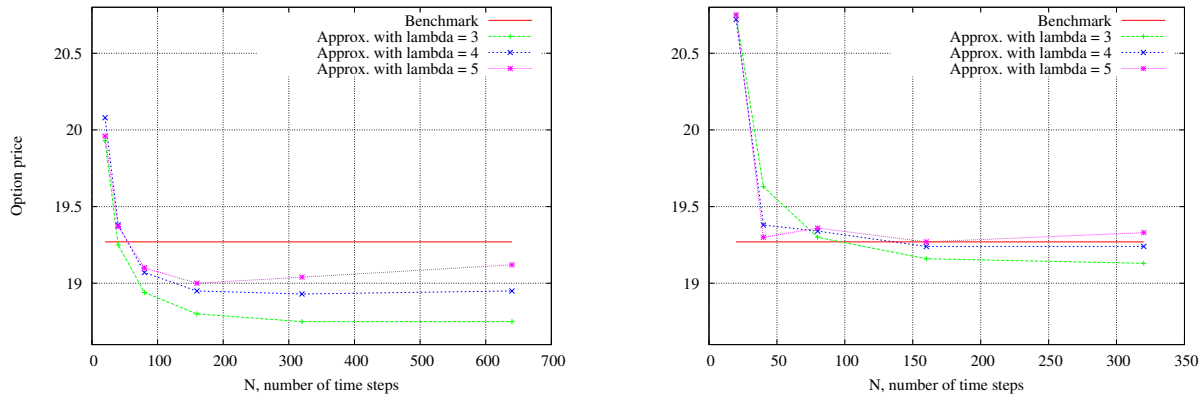

Figure 5.12: Approximate prices of a Swing option with 2 exercise rights and $\delta = \frac{1}{10}$, with $p = 5$ (left) and $p = 10$ (right).

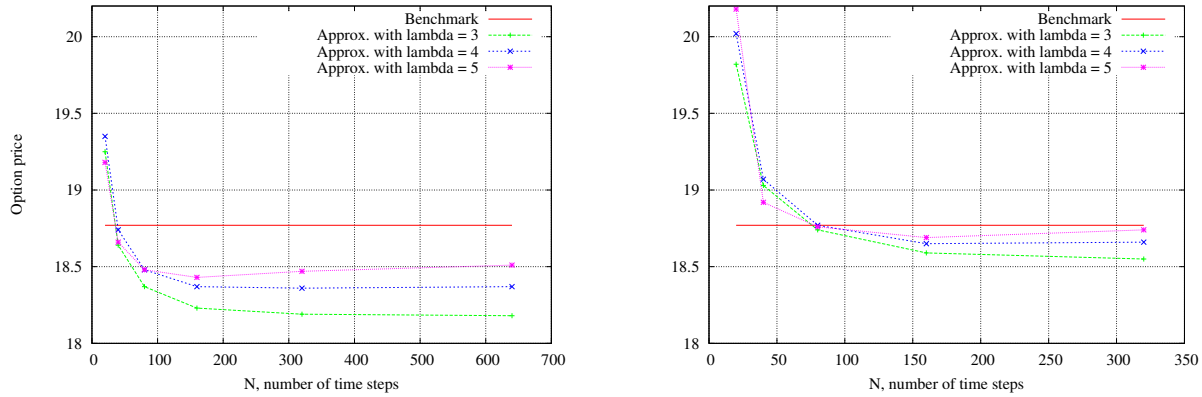

Figure 5.13: Approximate prices of a Swing option with 2 exercise rights and $\delta = \frac{2}{10}$, with $p = 5$ (left) and $p = 10$ (right).

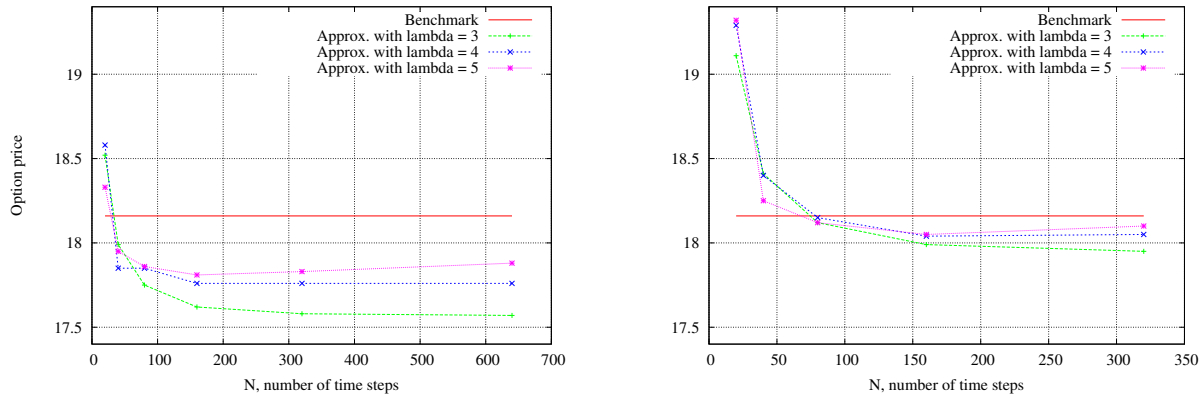

Figure 5.14: Approximate prices of a Swing option with 2 exercise rights and $\delta = \frac{3}{10}$, with $p = 5$ (left) and $p = 10$ (right).

For each considered value of λ, we retrieve a convergence in the number of time steps N of our method. As $p = 5$, approximate prices are converged from $N = 160$, so that we restrict ourselves to $N \leq 320$ time steps as $p = 10$. The limiting values are still below the benchmark but accurate option prices (relative error less than 1%) are obtained with a penalization coefficient p equal to 10 and $N = 160$. See also Table 5.4 in which the (signed) relative error to the benchmark is given in brackets. Besides, we observe a monotone convergence in λ of our approximate method.

	λ \ N	3	4	5
$\delta = \frac{1}{10}$	5	18.80 (-2.44%)	18.95 (-1.66%)	19.00 (-1.40%)
	10	19.16 (-0.57%)	19.24 (-0.16%)	19.27 (0.00%)
$\delta = \frac{2}{10}$	5	18.23 (-2.88%)	18.37 (-2.13%)	18.43 (-1.81%)
	10	18.59 (-0.96%)	18.65 (-0.64%)	18.69 (-0.43%)
$\delta = \frac{3}{10}$	5	17.62 (-2.97%)	17.76 (-2.20%)	17.81 (-1.93%)
	10	17.99 (-0.94%)	18.04 (-0.66%)	18.05 (-0.61%)

Table 5.4: Prices of a Swing option with 2 exercise rights (limiting values with $N = 160$).

We should point out that fine-tuning the parameters of the algorithm is difficult. As already mentioned, since the number of Monte Carlo paths is different in each set of sample paths $\mathcal{M}_{t_n}^q, q = 0, 1, 2$, the number of basis functions used for the least squares regressions has to be dynamically adapted. And when increasing much more the jump intensity λ, more Monte Carlo samples would be necessary.

For such a Swing option, the running time is much longer because the conditional expectations are computed by regression with respect to the bidimensional state variable (S^π, Θ^π). The computation of one option price takes at least 15 hours in above cases (when $N \geq 80$). In comparison, the benchmark method takes less than 5 minutes. Besides, the complexity of our method increases with $n_{\max}$, leading to untractable computational times for bigger values of $n_{\max}$, see Remark 5.2.3.

On this particular case of Swing options valuation, it seems that our method is less competitive than the classical approach. This is without any doubt due to the strong degeneracy of such a problem in our impulse control context: the valuation problem is 3-dimensional and involves an additional integer-valued state variable Q representing the number of exercise rights used at any considered time.

However, our method works and the numerical results that we obtain are consistent with the theoretical convergence rate given in Theorem 4.4.1.

Remark 5.2.3 (Dealing with more exercise rights)**.** The computational time of our method seems to increase *linearly* with the number of exercise rights $n_{\max}$. Indeed, at each time step of the backward induction procedure, the number of conditional expectation estimations is proportional to $n_{\max}$. Besides, when multiplying by 2 the number of exercise rights, it would require, at least, a double number of Monte Carlo samples for a same accuracy of the computation of conditional expectation estimators, see Remark 5.2.2.

Let us mention that the computational time of the *benchmark method using iteration*, described in Paragraph 5.2.2, increases linearly as function of the the maximal number of exercise

rights as well. We report in Figure 5.15 the price of a Swing option (with a fixed time-grid with $N = 50$ steps, a delay $\delta = \frac{1}{10}$, $M = 1$ million Monte Carlo paths and $b = 3$ basis functions) as function of the number of exercise rights $n_{\max}$ (left) and corresponding computational time (right).

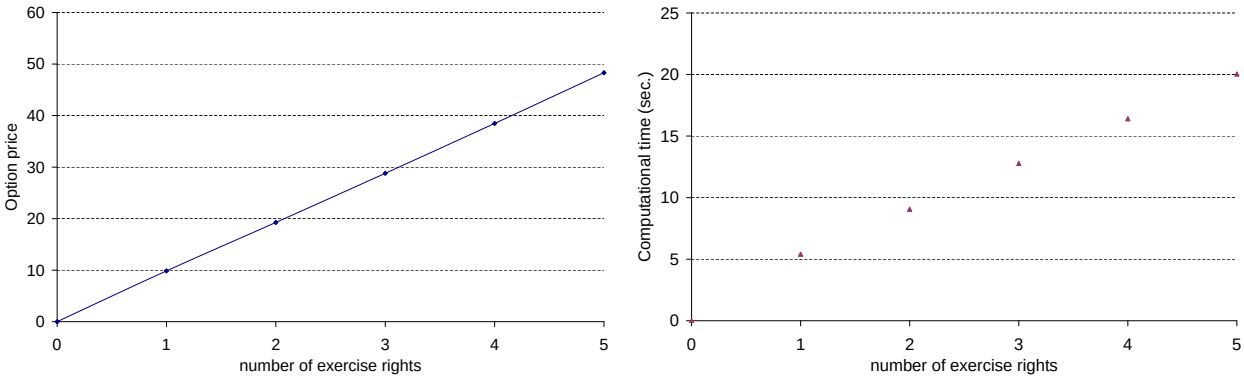

Figure 5.15: Price of a Swing option (left) and computational time required (right) as function of the number of exercise rights.

5.3 Valuation of gas storage facilities

For more details on the problem of gas storage valuation, we refer the reader to Chapter 1 in Part I: in particular, the problem is described in Section 1.2 and the storage value is solution of the optimal switching problem (1.6). The link to BSDEs with constrained jumps is justified in Paragraph 1.5.2.

Based on the same methodology developed for our case of impulse control, we present here a numerical scheme for valuing gas storage contract, which can be implemented via a *pure simulation-based method*. This algorithm provides an approximation at time 0 of the exact solution (1.6).

5.3.1 Solving the problem using BSDE with jumps

Let us assume that the gas spot price S process satisfies a one-factor mean-reverting gaussian model, namely:

$$\begin{cases} S_t = F(0,t)e^{-\frac{\sigma^2}{4a}(1-e^{-2at})+X_t} \\ dX_t = -aX_t dt + \sigma dW_t, \ X_0 = 0 \end{cases}$$

in which W is a standard Brownian motion, $(F(0,t))_{t\geq 0}$ a daily gas forward curve and a and σ constant mean-reverting and volatility coefficients respectively. The constrained BSDE with jumps associated to the storage valuation problem is:

$$\begin{cases} Y_t = g(S_T, C_T^I) + \int_t^T f(S_r, C_r^I, I_r)dr - \int_t^T Z_r dW_r - \int_t^T \int_{\mathcal{I}} V_r(j)\mu(dr,dj) + \int_t^T dK_r \\ -V_t(j) + \kappa(I_{t^-}, j) \geq 0, \forall j \in \mathcal{I} \end{cases} \tag{5.16}$$

in which μ is a Poisson random measure with intensity $\lambda(dj)dt$ and the forward processes (I, C^I) follow the dynamics:

$$\begin{aligned} dI_t &= \int_{\mathcal{I}} (j - I_{t^-}) \mu(dt, dj), \quad I_0 = i, \\ dC_t^I &= \bar{q}(t, C_t^I, I_t) dt, \quad C_0^I = c, \end{aligned}$$

where $\bar{q}$ is a modified version of the injection/withdrawal rate q, see (1.20), which makes the variable C^I satisfy the constraint[3]:

$$\underline{c}_t \leq C_t^I \leq \bar{c}_t.$$

We recall that I provides an artificial mode of operation and can be identified to a marked point process $(\mathrm{T}_k, \chi_k)_{k \geq 1}$ with $\mathrm{T}_0 = 0$ and $\chi_0 = i$:

$$I_t = \sum_{k \geq 0} \chi_k \mathbb{1}_{[\mathrm{T}_k, \mathrm{T}_{k+1})}(t), \quad \forall t \geq 0. \tag{5.17}$$

Let $p > 0$ be a penalization coefficient. The penalized BSDE with jumps associated to the storage valuation problem, which corresponds to constrained BSDE (5.16) in which the ($d\mathbb{P} \times dt \times \lambda(dj)$ almost sure) constraint on jumps penalizes the driver with a coefficient p, is:

$$\begin{aligned} Y_t^p = g(S_T, C_T^I) &+ \int_t^T \left[f(S_r, C_r^I, I_r) + p \int_{\mathcal{I}} \left(V_r^p(j) - \kappa(I_{r^-}, j) \right)^+ \lambda(dj) \right] dr \\ &- \int_t^T Z_r^p dW_r - \int_t^T \int_{\mathcal{I}} V_r^p(j) \mu(dr, dj). \end{aligned} \tag{5.18}$$

Under relevant assumptions (holding for the gas storage problem, up to the regularization of $\bar{q}$), if $K^p := p \int_0^{\cdot} \int_{\mathcal{I}} \left(V_r^p(j) - \kappa(I_{r^-}, j) \right)^+ \lambda(dj) dr$, then the solution $(Y^p, Z^p, V^p, K^p)_p$ to (5.18) tends to the solution (Y, Z, V, K) to (5.16) as p goes to infinity, see Elie and Kharroubi [49].

5.3.2 Discrete-time algorithm using Monte Carlo techniques

Let us introduce a regular time grid $\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}$ with N time steps. S can be approximated on π by a classical Euler scheme S^π. The pure jump process I can be simulated perfectly on $[0, T]$ (its jumps times are denoted by $(\mathrm{T}_k)_{k \geq 1}$, see (5.17)). As a consequence, an approximation of C^I on π, denoted by $C^{I,\pi}$, is given by:

$$\begin{cases} C_{t_0}^{I,\pi} = c \\ \forall t_{n+1} \in \pi : \\ \quad \text{if there is no jump on } (t_n, t_{n+1}] : \\ \quad\quad C_{t_{n+1}}^{I,\pi} = C_{t_n}^{I,\pi} + \bar{q}(t_n, C_{t_n}^{I,\pi}, I_{t_n}) \Delta t_{n+1} \\ \quad \text{else} : \\ \quad\quad \mathrm{T}_{k_n} := \text{ last jump time on } (t_n, t_{n+1}] \\ \quad\quad C_{t_{n+1}}^{I,\pi} = C_{t_n}^{I,\pi} + \bar{q}(t_n, C_{t_n}^{I,\pi}, I_{\mathrm{T}_{k_n}})(t_{n+1} - \mathrm{T}_{k_n}) \end{cases}$$

in which $\Delta t_{n+1} := t_{n+1} - t_n$. As the driver of the penalized BSDE with jumps (5.18) does not depend on Z^p, it is sufficient to compute $(Y^{p,\pi}, V^{p,\pi})$ on π, which can be made by the following

[3] Recall that $\underline{c}_t$ and $\bar{c}_t$ are given deterministice functions.

backward recursive scheme, see Remark 5.3.1:

$$\begin{cases} Y_{t_N}^{p,\pi} = g(P_{t_N}^{\pi}, C_{t_N}^{I,\pi}) \\ \forall t_n \in \pi, t_n < T : \\ \quad V_{t_n}^{p,\pi}(j) = \frac{1}{\lambda(j)\Delta t_{n+1}} \mathbb{E}_{t_n}\left[Y_{t_{n+1}}^{p,\pi} \tilde{\mu}\left((t_n, t_{n+1}] \times \{j\}\right)\right], \forall j \in \mathcal{I} \\ \quad Y_{t_n}^{p,\pi} = \mathbb{E}_{t_n}\left[Y_{t_{n+1}}^{p,\pi}\right] + f(S_{t_n}^{\pi}, C_{t_n}^{I,\pi}, I_{t_n})\Delta t_{n+1} \\ \qquad + \sum_{j\in\mathcal{I}} \left[p\left(V_{t_n}^{p,\pi}(j) - \kappa(I_{t_n}, j)\right)^+ - V_{t_n}^{p,\pi}(j)\right] \lambda(j)\Delta t_{n+1} \end{cases} \tag{5.19}$$

in which $\tilde{\mu}(dt, dj) = \mu(dt, dj) - \lambda(dj)dt$ is the compensated measure associated to μ and:

$$\mathbb{E}_{t_n}[\cdot] := \mathbb{E}\left[\cdot | \left(S_{t_n}^{\pi}, C_{t_n}^{I,\pi}, I_{t_n}\right)\right]. \tag{5.20}$$

The discretization error of this scheme tends to zero as $|\pi| := \Delta t_{n+1}$ goes to 0, by the same arguments as Bouchard and Elie [19].

Remark 5.3.1 (Intuition for the backward numerical scheme (5.19))**.** We apply the same arguments as Bouchard and Elie [19] in this specific context of optimal switching, where the intensity measure λ has a discrete support, namely $\mathcal{I} = \{-1, 0, +1\}$. From the terminal condition of (5.18), we have $Y_{t_N} = g(S_{t_N}^{\pi}, C_{t_N}^{I,\pi})$. Writing the BSDE (5.18) on $[t_n, t_{n+1})$ and using an explicitation at time t_n, we get

$$\begin{aligned} Y_{t_n}^{p,\pi} = Y_{t_{n+1}}^{p,\pi} + \big[\ f(S_{t_n}^{\pi}, C_{t_n}^{I,\pi}, I_{t_n}^{\pi}) + p\sum_{j\in\mathcal{I}} \left(V_{t_n}^{p}(j) - \kappa(I_{t_n}, j)\right)^+ \lambda(j)\ \big]\Delta t_{n+1} \\ - Z_{t_n}^{p,\pi}\Delta W_{t_{n+1}} - \sum_{j\in\mathcal{I}} V_{t_n}^{p,\pi}(j)\mu\left((t_n, t_{n+1}] \times \{j\}\right). \end{aligned} \tag{5.21}$$

By multiplying (5.21) by $\tilde{\mu}\left((t_n, t_{n+1}] \times \{j\}\right), \forall j \in \mathcal{I}$ and taking conditional expectation $\mathbb{E}_{t_n}[\cdot]$,

$$V_{t_n}^{\pi}(j) = \frac{1}{\lambda(j)\Delta t_{n+1}} \mathbb{E}_{t_n}\left[Y_{t_{n+1}}^{p,\pi} \tilde{\mu}\left((t_n, t_{n+1}] \times \{j\}\right)\right], \forall j \in \mathcal{I},$$

since $\mathrm{covar}_{t_n}(\tilde{\mu}\left((t_n, t_{n+1}] \times \{k\}\right), \tilde{\mu}\left((t_n, t_{n+1}] \times \{j\}\right)) = \delta_{k,j}\lambda(j)\Delta t_{n+1}$, $\forall (k, j) \in \mathcal{I}^2$. Finally, by taking conditional expectation $\mathbb{E}_{t_n}[\cdot]$ in (5.21) we retrieve the expression for $Y_{t_n}^{\pi}$ in (5.19) by $\mathbb{E}_{t_n}[\mu\left((t_n, t_{n+1}] \times \{j\}\right)] = \lambda(j)\Delta t_{n+1}$.

Monte Carlo based resolution

We are now ready to provide a purely simulation-based method giving an approximation of the exact solution (1.6) at time 0. The last task consists in estimating the conditional expectations operators $\mathbb{E}_{t_n}$, recall (5.20). We shall use the same least squares Monte Carlo approach as above, based on the improved technique of adaptative local basis proposed by Bouchard and Warin [21]. However, since this regression-based method is only relevant for *real-valued variables* (the regression basis functions have compact support), it cannot handle with the $\{-1; 0; 1\}$-valued variable I and we need to adapt the Monte Carlo scheme in consequence.

Given $M \geq 1$ i.i.d. Monte Carlo samples of forward variables

$$\left(S^{\pi,(m)}, C^{I,\pi,(m)}, I^{(m)}\right), \forall m \leq M$$

we can overcome this difficulty by separating, at each time step $t_n < T$, the set of M Monte Carlo paths with respect to the value of $I^{(m)}$. That is, introduce three sets:

$$\mathcal{M}_{t_n}^k := \left\{ m = 1, \dots, M : I_{t_n}^{(m)} = k \right\}, \quad \forall k \in \mathcal{I} = \{-1; 0; 1\}.$$

which forms a partition of $\{1, \dots, M\}$ and finally compute the solution on each Monte Carlo path, namely

$$\left(Y_{t_n}^{p,\pi,(m)}, V_{t_n}^{p,\pi,(m)} \right)_{m \leq M} \quad \forall t_n \in \pi$$

by the following fully implementable backward algorithm:

1. Initialization: $Y_{t_N}^{p,\pi,(m)} = g(S_{t_N}^{\pi,(m)}, C_{t_N}^{I,\pi,(m)}), \forall m \leq M$.
2. Backward induction for $n = N-1, \dots, 0$:

   For any $k \in \mathcal{I} = \{-1; 0; 1\}$, and $m \in \mathcal{M}_{t_n}^k$, conditional expectations estimations

   $$\begin{aligned} \varepsilon_{t_n}^{V,k,(m)}(j) &\approx \mathbb{E}\left[Y_{t_{n+1}}^{p,\pi} \tilde{\mu}\left((t_n, t_{n+1}] \times \{j\} \right) | \left(S_{t_n}^{\pi,(m)}, C_{t_n}^{I,\pi,(m)}, I_{t_n} = k \right) \right], \forall j \in \mathcal{I} \\ \varepsilon_{t_n}^{Y,k,(m)} &\approx \mathbb{E}\left[Y_{t_{n+1}}^{p,\pi} | \left(S_{t_n}^{\pi,(m)}, C_{t_n}^{I,\pi,(m)}, I_{t_n} = k \right) \right] \end{aligned}$$

   are respectively provided by least squares regression of

   $$\left(Y_{t_{n+1}}^{p,\pi,(m)} \tilde{\mu}^{(m)}\left((t_n, t_{n+1}] \times \{j\} \right) \right)_{m \in \mathcal{M}_{t_n}^k} \text{ and } \left(Y_{t_{n+1}}^{p,\pi,(m)} \right)_{m \in \mathcal{M}_{t_n}^k}$$

   on

   $$\left(\psi_1(S_{t_n}^{\pi,(m)}, C_{t_n}^{I,\pi,(m)}), \dots, \psi_b(S_{t_n}^{\pi,(m)}, C_{t_n}^{I,\pi,(m)}) \right)_{m \in \mathcal{M}_{t_n}^k}$$

   where $(\psi_l)_{l=1,\dots,b}$ are given basis functions. Then,

   $$\begin{cases} V_{t_n}^{p,\pi,(m)}(j) &= \frac{1}{\lambda(j)\Delta t_{n+1}} \varepsilon_{t_n}^{V,k,(m)}(j), \forall j \in \mathcal{I}, \\ Y_{t_n}^{p,\pi,(m)} &= \varepsilon_{t_n}^{Y,k,(m)} + f(S_{t_n}^{\pi,(m)}, C_{t_n}^{I,\pi,(m)}, k)\Delta t_{n+1} \\ & \quad + \sum_{j \in \mathcal{I}} \left[\, p\left(V_{t_n}^{p,\pi,(m)}(j) - \kappa(k,j) \right)^+ - V_{t_n}^{p,\pi,(m)}(j) \, \right] \lambda(j) \Delta t_{n+1}. \end{cases}$$

3. Estimation of the storage value at time 0 (recall $\mathcal{M}_{t_0}^i = \{1, \dots, M\}$ and $\mathcal{M}_{t_0}^k = \emptyset, \forall k \neq i$ since the initial operating mode is $u_0 = I_0 = i$):

   $$\begin{cases} V_{t_0}^{p,\pi}(j) &= \frac{1}{\lambda(j)\Delta t_1} \sum_{m=1}^M \left(Y_{t_1}^{p,\pi,(m)} \tilde{\mu}^{(m)}\left((t_0, t_1] \times \{j\} \right) \right), \forall j \in \mathcal{I}, \\ Y_{t_0}^{p,\pi} &= \frac{1}{M} \sum_{m=1}^M Y_{t_1}^{p,\pi,(m)} + f(s, c, i)\Delta t_1 \\ & \quad + \sum_{j \in \mathcal{I}} \left[\, p\left(V_{t_0}^{p,i,\pi,(m)}(j) - \kappa(i,j) \right)^+ - V_{t_0}^{p,i,\pi,(m)}(j) \, \right] \lambda(j) \Delta t_1. \end{cases}$$

5.3.3 Numerical observations and improvement of the method

We notice that the choice of the intensity measure of the Poisson measure μ has a critical impact on the dynamics of the forward process C^I, which stands for the artificial inventory level, governed by the mode of operation I. Recall the definition of I as a marked point process $(\mathrm{T}_k, \chi_k)_{k \geq 0}$ in

(5.17). A first natural (but naive) choice consists in simulating this artificial operation regime I with equiprobable marked values in $\mathcal{I}$:

$$\chi_k \in \mathcal{U}\left(\{-1;0;+1\}\right), \quad \forall k \geq 1,$$

the $(\mathrm{T}_k)_{k\geq 1}$ being the jump times of a Poisson process with intensity $\bar{\lambda}$. Then, as $\bar{\lambda}$ increases, the corresponding trajectories of $C^{I,\pi}$ narrow: see Figure 5.16 for a peak load storage facility (with storage characteristics $\underline{c}_t = 0, \overline{c}_t = 400$, $q_{\text{inj}} = q_{\text{with}} = 20$: 20 days are necessary to totally fill or empty the facility) and Figure 5.17 for a base load storage facility (with $\underline{c}_t = 0, \overline{c}_t = 1000$, $q_{\text{inj}} = q_{\text{with}} = 5$: 200 days necessary to totally fill or empty the facility).

In consequence, the set of reached inventory levels is strongly dependent on the choice of $\bar{\lambda}$: in particular, for too large values $\bar{\lambda}$, the whole set of admissible inventory will not be crossed by sample paths of C^I.

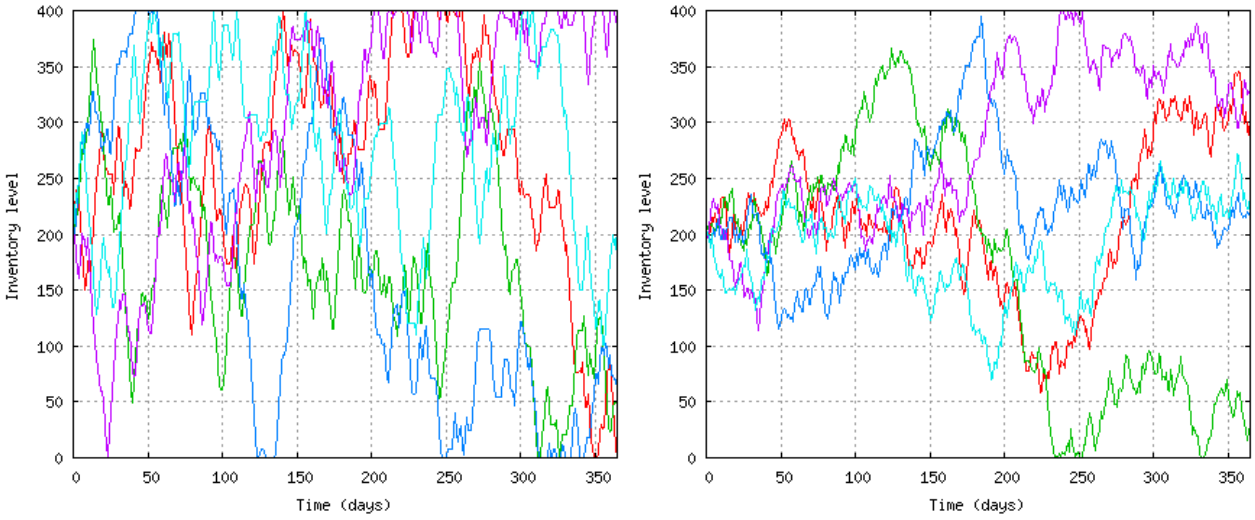

Figure 5.16: Peak load storage: trajectories of C^I with $\bar{\lambda} = 1$ (left) and $\bar{\lambda} = 5$ (right).

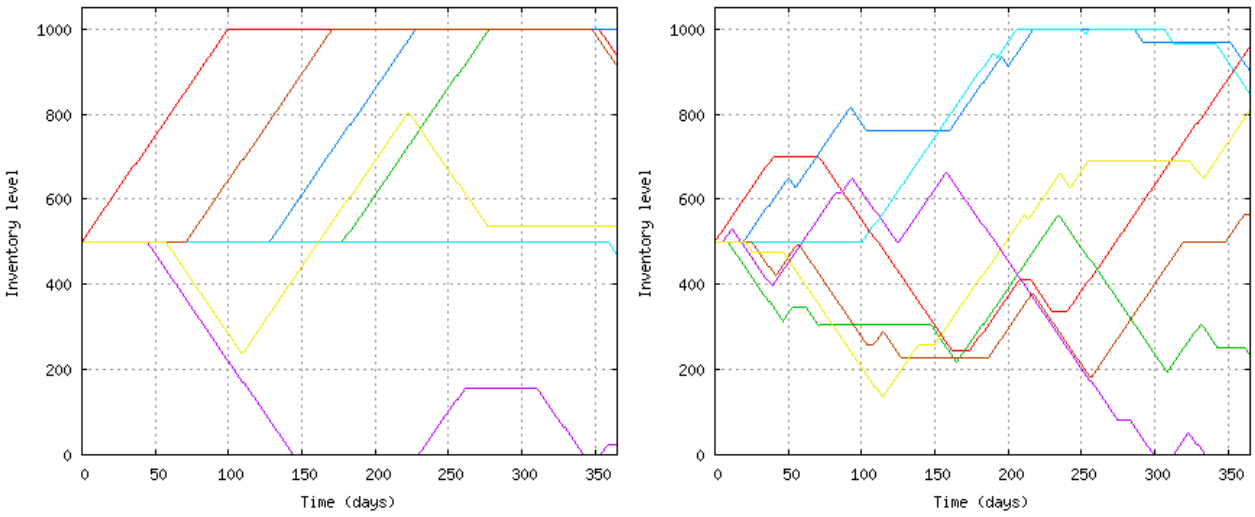

Figure 5.17: Base load storage: trajectories of C^I with $\bar{\lambda} = 0.01$ (left) and $\bar{\lambda} = 0.05$ (right).

An improvement of the simulation procedure of I is possible. It consists in weighting *heuristically* one mode or the other, depending on the seasonal trend of the initial gas forward curve $(F(0,t))_{t\geq 0}$, see Figure 5.18[4]. In other words, one can assign to the regime $k = +1$, that is injection, a stronger probability weight in summer and/or at the end of every week (lower trend of gas prices), and to the regime $k = -1$ (withdrawal) a stronger probability weight in winter and/or at the beginning of every week (higher trend of gas prices).

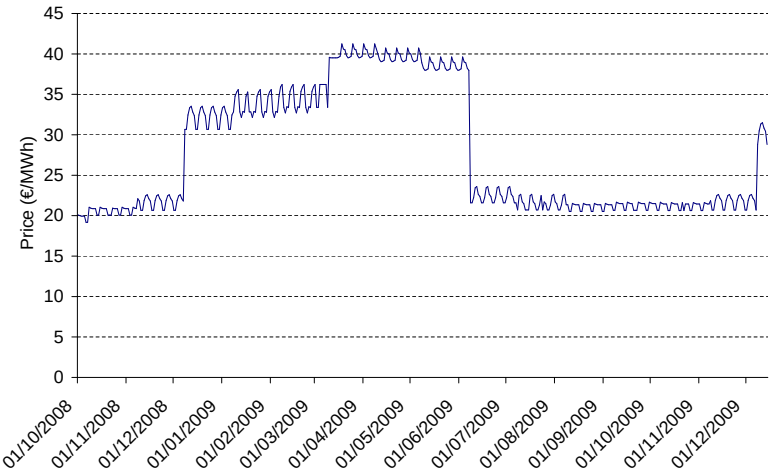

Figure 5.18: Gas daily forward curve.

Basically, this allows to force the inventory level sample paths to cross the level regions in the cavity, which will be *most likely crossed by the optimal strategy*.

For the base load facility described above for example, this heuristic rule forces to fill the cavity in summer and empty it in winter. We report in Figure 5.19 trajectories of the gas price S^5 (left) and of the artificial inventory level C^I (right) when weighting mode $+1$ in summer (forward prices ≤ 27 €/MWh) and -1 in winter (forward prices > 27 €/MWh) with a probability $p_{\max} = 0.8$ (the two other modes being weighted with $p_{\min} = 0.1$).

[4]Historical prices between October 2008 and December 2009 observed on the Zeebrugge gas market.
[5]Coefficients of the gas spot price model: $a = 1.5$, $\sigma = 59$.

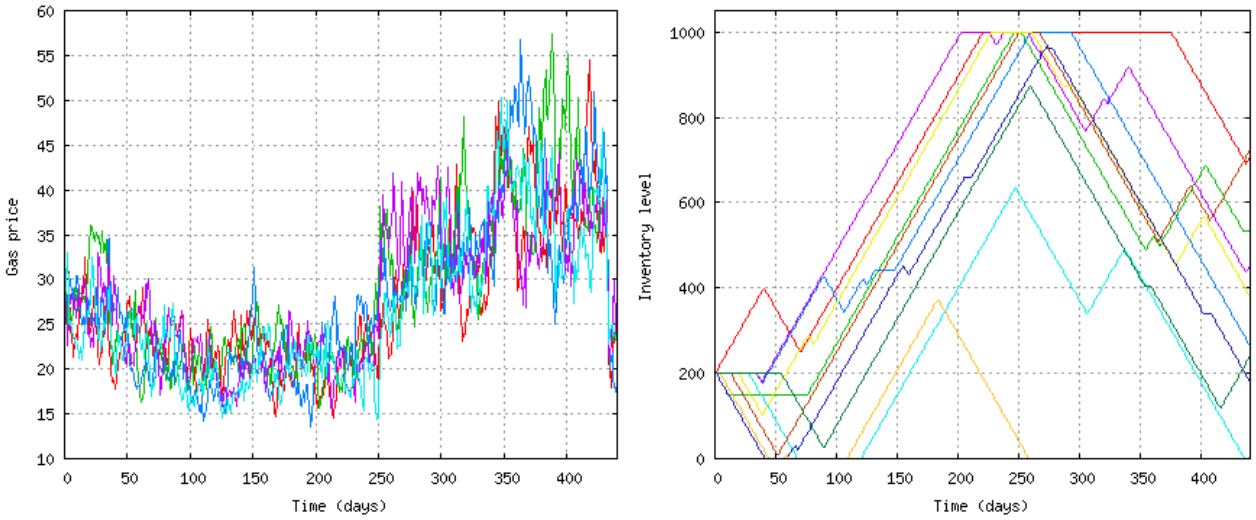

Figure 5.19: Base load storage: trajectories of (S, C^I) with $\bar{\lambda} = 0.05$ and $p_{\max} = 0.8$.

For a peak load facility, this kind of heuristic should take into account the intra-week seasonality of the gas forward curve as well. We have to highlight that this kind of heuristic consists in fact in fitting the intensity measure λ, with respect to the *a priori optimal behavior of the storage*, in other words to the operational characteristics of the storage facility (withdrawl/injection rates and volumetric constraints). In consequence, such a method cannot be generic.

To be efficient, the numerical algorithm presented in Paragraph 5.3.2 needs to be modified with the above described heuristic for simulating the artificial regime of operation I. The achievement of gas storage valuation results with such a method remains a challenging task.

Chapter 6

Concluding remarks and perspectives

Penalization approach for solving impulse control problems by using BSDEs with jumps The penalization procedure that we introduce seems to constitute an efficient method when dealing with non degenerate forward processes, cf. the optimal forest management problem considered in Section 5.1. This allows to avoid classical methods, based on an iteration on the number of interventions.

However, for the valuation of Swing options, see Section 5.2, the complexity of our numerical method increases linearly with the number of exercise rights $n_{\max}$ as any classical iteration-based methods. This is due to the fact that we need to consider an additional discrete-valued state variable, standing for the (artificial) number of exercise rights used at any considered time. At each time step of the backward scheme, we thus perform a conditional expectation estimation for each possible value of this variable (that is equal to $0, 1, \ldots, n_{\max}-1$ or $\geq n_{\max}$), see Remark 5.2.3. Our method requires a large computational effort (slow computation due to a large number of Monte Carlo samples necessary) and cannot be competitive in comparison to more classical techniques.

Statistical error introduced by the least squares Monte Carlo approach A control of the statistical error introduced by the least squares Monte Carlo (LSM) approach is provided in Gobet et al. [56] (see Lemor [80] for further details). By extension, this applies to BSDE with jumps (see Elie [48]) to control the error on the jump component $V^{p,\pi}$ and thus ensures that the least squares Monte Carlo error tends to 0 as the number of samples M and the number of basis functions b tends to $+\infty$.

We have numerically observed that the variance of our Monte Carlo method grows when increasing the jump intensity λ and the penalization parameter p and our intuition is that such a statistical error might exponentially increase with (λp^2), similarly to the error due to discretization. This interesting question remains open and one could use the same arguments as Lemor [80] to derive an explicit control of the LSM approximation error with respect to approximation parameters (λ, p, M, b).

Reconstruction of the optimal strategy We should point out that our penalization-based numerical method does not provide any information about the optimal switching strategy during the backward in time recursion. This constitutes a major drawback of our approach. An additional numerical procedure, to be determined, would be required.

Dealing with more general impulse control problems From a theoretical viewpoint, one could consider the more general class of impulse control problems, where strategies also involve a sequence of impulses exercised at the corresponding intervention times: that is, impulse strategies are double sequences $u = (\tau_k, \xi_k)_{k\geq 1}$ in which ξ_k is a $\mathcal{F}_{\tau_k}$-measurable variable valued in some compact set E. Our penalization approach still holds (see [71]) and a challenging question concerns the extension of our convergence rate estimations to this more general framework.

The same arguments as the one that we introduce for computing a convergence rate of the penalization error in Chapter 3 might be used. First, the $\frac{1}{2}$-Hölder property of the value function still holds, see Proposition 2.3.1. In addition, our key argument of our reasoning seems to be applicable. One might approximate the solution to

$$v_T^\eta(0,x) = \sup_{u=(\tau_k,\xi_k)_{k\geq 1}\in\mathcal{U}_{(0,T-\eta]}} \mathbb{E}\left[g(X_T^u) + \int_t^T f(X_s^u)ds + \sum_{k\geq 1, t<\tau_k\leq T} \kappa(X^u_{\tau_k^-}, \xi_k)\right] \tag{6.1}$$

by:

$$\tilde{Y}_0^p = \mathbb{E}\left[g(X_T^p) + \int_0^T f(X_t^p)dt + \int_0^T \int_E \kappa(X^p_{t^-}, e)\mu^p(dt, de)\right]$$

in which X^p is a forward process whose jumps are governed by some random measure μ^p with intensity $\lambda(de)\nu_t^p(e)dt$, for a suitable choice of $(\nu_t^p(e))_{e\in E}$ with respect to the optimal impulse strategy $(\tau_k^*, \xi_k^*)_{k\geq 1}$ to (6.1): it will intuitively force the penalized solution to jump as soon as possible after that an optimal intervention happens with a size having a distribution resembling to the distribution of the optimal impulse. This last feature remains a challenging point, open for further research.

On the other hand, the estimation that we performed for the discretization error still holds with the same arguments as [48]. One has just to replace λ by $\lambda(E)$ in the final convergence rate, see Corollary 4.4.1.

Perspectives for storage facilities valuation We have provided a purely simulation-based algorithm for the valuation of gas storage facilities. An improvement for the *intensity measure choice* is necessary: we propose a method which forces the inventory level sample paths to cross the level regions in the cavity, which will be most likely crossed by the optimal strategy, see Paragraph 5.3.3. The achievement of gas storage valuation results with such a method remains a challenging task.

Our intuition, however, is that such a method could not be competitive with other existing valuation methods. A too large computational time is required and it cannot be generic with respect to the kind of storage facility considered.

Part III

Valuation methods for moving average options

Chapter 1

Introduction

We consider American-style financial derivatives whose payoff at exercise depends on the average(s) of underlying asset price(s) over a rolling period in time. These kind of options will be called *moving average options*: they are particularly complex because they do not only incorporate an early exercise feature, but are path-dependent.

American-style options on moving average prices are principally used in corporate finance and in energy markets. The common point of these uses is that for many investors, the moving average is a popular technical measure for trends or trend reversals of market prices. This indicator is in particular frequently used in technical analysis. The simplest example (sometimes known as surge option) is a variable strike call or put, whose strike is adjusted daily to the moving average of the underlying asset over a certain fixed-length period preceding the current date. Moving average options are widely used in energy markets. In gas markets, for example, these options are known as *indexed Swing options*. These supplying contracts allow the holder to purchase an amount of gas at a strike price, which is indexed on moving averages of various oil-prices: typically gas oil and fuel oil prices are averaged over the last 6 months and delayed in time with a 1 month lag.

We shall denote by X the moving average of an underlying S over a time window with fixed length $\delta > 0$:

$$X_t = \frac{1}{\delta} \int_{t-\delta}^{t} S_u du, \quad \forall t \geq \delta.$$

The process X follows the dynamics

$$dX_t = \tfrac{1}{\delta} \left(S_t - S_{t-\delta} \right) dt, \quad \forall t \geq \delta.$$

This shows in particular that even if S is Markovian, the process (S, X) is not: it is, in general, impossible for any finite n to find n processes $X^1, \ldots, X^n$ such that $(S, X, X^1, \ldots, X^n)$ are jointly Markovian. This property makes the pricing of the moving window options with early exercise a challenging problem both from the theoretical and the numerical viewpoint.

In a continuous-time framework, the problem is *infinite dimensional* since the cashflow from exercise depends on the path of the underlying asset price over the averaging window. In a discrete-time framework (pricing of a Bermudan option instead of an American option) there is a computational challenge, due to *high dimensionality*: the dimension is equal to the number of time steps within the averaging window, and in presence of a time delay, it is increased by the

number of time steps within the lag period. This in particular makes it difficult to compute the conditional expectations involved in the optimal exercise rule.

The problem of pricing moving average American options should not be confused with a much simpler problem of pricing Asian American options with a fixed start averaging window, where the payoff depends on

$$A_t = \frac{1}{t}\int_0^t S_u du, \forall t > 0.$$

It is well-known (see for example Wilmott and al. [114]) that in this case, adding a dimension to the problem allows to derive a finite-dimensional Markovian formulation.

On the other hand, partial average Asian options of European style can be easily valued (see for example Shreeve [105]). If the averaging period has a length $\delta > 0$, then on $[T-\delta, T]$ the option value is given by the price of the corresponding Asian option and on $[0, T-\delta]$ it solves a European style PDE with appropriate terminal and boundary conditions.

In the literature, very few articles discuss moving average options with early exercise feature [22, 15, 57, 68, 38]. In a discrete-time setting, a common approach (see e.g., Broadie and Cao [22]) is to use the least squares Monte Carlo, computing the conditional expectation estimators through regressions on polynomials of the current values of the underlying price and its moving average, which constitutes a non Markovian approximation. Since the future evolution of the moving average depends on the entire history of the price process between $t-\delta$ and t, this approach introduces a bias, leading to a suboptimal price. This kind of non Markovian approximation is widely used by practitioners: in particular, for valuing oil-indexed Swing options in which the averaging window is very large, commonly 5 or 6 months (recall the gas market provides daily spot prices). And yet, no theoretical result justifies this approximation.

Otherwise, to our best knowledge, all the other existing numerical approaches are computationally limited to applications where the moving average window is small. Bilger [15] uses the Longstaff and Schwartz's algorithm (see Longstaff and Schwartz [82]) for the valuation of moving window Bermudan options. The author uses a regression based approach to compute the conditional expectations considering that the state vector is composed of the underlying price, its moving average and additional partial averages of the price over the rolling period. Their number is computed heuristically and as it tends to the number of time steps within the rolling period, the computed price tends to the true price of the moving average option. The same kind of approach is used by Grau [57], but the author improves its numerical efficiency by a different choice of basis functions in the regressions used for the conditional expectations estimation. Its computational resources allow him to deal with numerical experiments up to dimension 10.

Kao and Lyuu [68] propose a tree method based on the CRR model to price moving average lookback and reset options. Their method can handle only short averaging windows: the numerical results that are shown deal at most with 5 discrete observations in the averaging period. Indeed, this tree-based approach leads to an algorithm complexity (number of tree nodes) which exponentially increases with the number of time steps in the averaging period. Finally, Dai et al. [38] introduce a lattice algorithm for pricing Bermudan moving average barrier options. The authors propose a finite-dimensional PDE model for such options and solve it using a grid method.

The pricing of moving average options is closely related to high-dimensional optimal stopping problems. It is well-known that deterministic techniques such as finite differences or approximating trees are made inefficient by the so-called curse of dimensionality. Only Monte Carlo type techniques can handle American options in high dimensions. Bouchard and Warin [21] and

references therein shall give to the interested reader a recent review of this research field. In [21], some improvements are proposed for the estimation of conditional expectations, which allow to deal with state vectors up to dimension 8.

More generally, in a continuous-time setting, a related problem is that of optimal stopping of stochastic differential equations with delay. With the exception of a few cases where explicit dimension reduction is possible [51, 54], there is no numerical method for solving such problems.

We propose a method for pricing moving average American options based on a finite dimensional approximation of the infinite-dimensional dynamics of the moving average process. The approximation is based on a truncated expansion of the weighting measure used for averaging in a series involving Laguerre polynomials. This technique has long been used in signal processing for the approximation of infinite-dimensional systems (see Lee [79] for an early reference on the subject and Mäkilä [89] for a more recent one), but is less known in the context of approximation of stochastic systems.

The resulting problem is then a finite-dimensional optimal stopping problem, which we propose to solve with a Monte Carlo Longstaff and Schwartz-type approach. Such a numerical method has many advantages: it is independent of the underlying price model, still competitive when adding a time lag or in a multi-assets framework and the error made does not increase as the averaging time window length does. In comparison to existing methods whose dimension is conditioned by the number of time steps within the averaging period and the time lag (which might be large in practical cases), this method allows to *uniformly and roughly reduce the state dimension.*

The rest of this Part is structured as follows. In Chapter 2, we introduce the mathematical context and formulate the stochastic control problem related to pricing of moving average American options with time delay. We provide a general result which links the strong error of approximating one moving average process with another to a certain distance between their weighting measures. We then introduce an approximation of the weighting measure as a series of Laguerre functions truncated at n terms, which leads to $(n+1)$-dimensional Markovian approximation to the initial infinite-dimensional problem. The properties of Laguerre functions combined with our strong approximation result then enable us to establish a bound on the pricing error introduced by our approach as n goes to infinity, as soon as the option payoff is Lipschitz in the moving average variable.

In Chapter 3, we present our numerical method, based first on the optimal scaling of the Laguerre functions introduced in the approximation and then on the least squares Monte Carlo algorithm for the discrete-time resolution of the optimal stopping problem. The implementation through Monte Carlo techniques is very easy. In particular, all the Laguerre-based approximations can be explicitly computed. We then introduce a benchmark method and the above mentioned non Markovian approximate method, which will be used as a lower reference when the dimension is too high for using the benchmark (both are based on the least squares Monte Carlo algorithm as well).

In Chapter 4, we report two set of numerical applications. Firstly, we present the results of numerical experiments in the Black-Scholes framework which include pricing moving average options with time delay. Secondly, we provide the pricing results obtained for oil-indexed American-style options in the European gas market.

We report in the final chapter some directions for further research.

Chapter 2

A finite-dimensional approximation for pricing moving average options

2.1 Framework and formulation of the pricing problem

Let us consider an underlying asset whose price $S = (S_t)_{t\geq 0}$ is a $\mathbb{R}$-valued non-negative Markov process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{P}$ is a martingale probability for the financial market[1] and $\mathbb{F} = (\mathcal{F}_t)_{t\leq T}$ denotes the natural filtration generated by S up to a fixed time horizon T. For the sake of simplicity, we present our results in the framework of a one-dimensional price model but they are directly generalizable to a multi-asset model, see Remark 2.1.1, or to a model with unobservable risk factors such as stochastic volatility.

We shall denote by X the moving average of S over a time window with fixed length $\delta > 0$ delayed with a fixed time lag $l \geq 0$:

$$X_t = \frac{1}{\delta}\int_{t-l-\delta}^{t-l} S_u du, \quad \forall t \geq \delta + l. \tag{2.1}$$

The moving average American option pricing problem that we study is the following:

$$\sup_{\tau\in\mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, X_\tau\right)\right] \tag{2.2}$$

in which $\mathcal{T}_{[\delta+l,T]}$ is the set of $\mathbb{F}$-stopping times valued in $[\delta + l, T]$ and $\phi : \mathbb{R}_+^2 \to \mathbb{R}$ a payoff function. We will use the assumption:

(A1) ϕ is Lipschitz in its second variable.

In addition, we shall adopt the following convention for the values of S on the negative time-axis:

$$S_t = S_0, \quad \forall t \leq 0. \tag{2.3}$$

As already mentioned in the introduction, the problem (2.2) is an *infinite dimensional* optimal stopping problem. X defined in (2.1) is a standard equally weighted moving average, with dynamics:

$$dX_t = \tfrac{1}{\delta}\left(S_{t-l} - S_{t-l-\delta}\right)dt, \quad \forall t \geq \delta + l.$$

[1] Without loss of generality, we set the interest rate to zero.

This shows in particular that the process (S, X) is not Markovian: it is, in general, impossible for any finite n to find n processes $X^1, \ldots, X^n$ such that $(S, X, X^1, \ldots, X^n)$ are jointly Markov. In contrast to the above, the process (S, A) most relevant in pricing Asian options in which:

$$A_t = \frac{1}{t}\int_0^t S_u du, \forall t > 0$$

is Markovian because:

$$dA_t = \tfrac{1}{t}\left(S_t - A_t\right) dt, \forall t > 0.$$

Remark 2.1.1 (Case of a multi-asset model)**.** Let $(S^j)_{j=1,\ldots,d}$ be the Markovian price processes of d assets, $\mathcal{J}$ a set of indexes in $\{1, 2, \ldots, d\}$ and a moving average defined by

$$X_t = K + \sum_{j\in\mathcal{J}} \alpha_j \left[\frac{1}{\delta}\int_{t-l-\delta}^{t-l} S_u^j du\right], \quad \forall t \geq \delta + l,$$

in which $K \geq 0$ is a fixed price and $(\alpha_j)_{j\in\mathcal{J}}$ real positive constants corresponding to the weights assigned to each moving average composing the index X. One can then consider a payoff function $\phi : \mathbb{R}_+^{d+1} \to \mathbb{R}$ and the option pricing problem:

$$\sup_{\tau\in\mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau^1, S_\tau^2, \ldots, S_\tau^d, X_\tau\right)\right].$$

Our results presented for the one-dimensional case are directly generalizable to this multi-asset framework. This includes in particular the problem of indexed options described in the introduction.

Notations We will denote by $\mathrm{L}^2 := \mathrm{L}^2([0, +\infty))$ the Lebesgue space of real-valued square-integrable functions f on $[0, +\infty)$ endowed with its norm:

$$\|f\|_2 := \left[\int_0^\infty |f(x)|^2 dx\right]^{\frac{1}{2}}$$

and by $\langle\cdot,\cdot\rangle$ the associated scalar product. Besides, we will use the classical Landau symbol $f_n = \mathcal{O}(g_n)$ meaning that $|f_n| \leq C g_n$ for some constant $C > 0$.

2.2 A finite-dimensional approximation of moving average options price

The moving average X in (2.1) can be rewritten (by a straightforward change of time variable) as:

$$X_t = \int_0^\infty S_{t-u}\left(\frac{1}{\delta}\mathbb{1}_{[l,l+\delta]}(u)\right) du.$$

In the sequel, we consider a moving average with a more general weighting function, that is a moving average process of the form[2]:

$$M_t = \int_0^\infty S_{t-u}\mu(du), \tag{2.4}$$

where μ is a finite possibly signed measure on $[0, \infty)$, recall convention (2.3).

[2] In the literature (see Basse and Pedersen [11] and references therein), moving averages are usually defined via the stochastic integral of S. Our definition as an ordinary integral with respect to a weighting measure is closer to the financial specifications.

2.2.1 Strong approximations of moving average processes

The following Lemma 2.2.1 provides a tool for comparing two moving averages M and N on S with different weighting measures with respect to the distance:

$$\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right].$$

We shall use an integrability assumption on the modulus of continuity of the price process being averaged S:

(A2) There exists a constant $C < \infty$ such that

$$\mathbb{E}\left[\sup_{t,s\in[0,T]:|t-s|\le h}|S_t - S_s|\right] \le C\varepsilon(h), \quad \varepsilon(h) := \sqrt{h\ln\left(\frac{2T}{h}\right)}.$$

Fisher and Nappo [53] show that Assumption (A2) holds in particular when S is a continuous Itô process of the form

$$S_t = S_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s$$

with

$$\mathbb{E}\left[\sup_{0\le t\le T}|b_s|\right] < \infty \quad \text{and} \quad \mathbb{E}\left[\sup_{0\le t\le T}|\sigma_s|^{1+\gamma}\right] < \infty$$

for some $\gamma > 0$.

Lemma 2.2.1. Let Assumption (A2) be satisfied, let μ and ν be finite signed measures on $[0,\infty)$ with Jordan decompositions $\mu = \mu^+ - \mu^-$ and $\nu = \nu^+ - \nu^-$, such that $\mu^+(\mathbb{R}_+) > 0$. Define

$$M_t = \int_0^\infty S_{t-u}\mu(du), \qquad N_t = \int_0^\infty S_{t-u}\nu(du).$$

Then

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] &\le C|\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)| \\ &+ C\Big(\mu^+([0,T]) + \nu^-([0,T]) + |\mu([0,T]) - \nu([0,T])|\Big)\,\varepsilon\left(\frac{1}{\mu^+([0,T])}\int_0^T |F_\mu(t) - F_\nu(t)|dt\right)\end{aligned} \tag{2.5}$$

for some constant $C < \infty$ which does not depend on μ and ν, where

$$F_\nu(t) := \nu([0,t]) \quad \text{and} \quad F_\mu(t) := \mu([0,t]).$$

Proof. *Step 1.* We first assume that μ and ν are probability measures. Let F_μ^{-1} and F_ν^{-1} be generalized inverses of μ and ν respectively. Then,

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] &= \mathbb{E}\left[\sup_{0\le t\le T}\int_0^1 |S_{t-F_\mu^{-1}(u)} - S_{t-F_\nu^{-1}(u)}|du\right] \\
&\le \int_0^1 \mathbb{E}\left[\sup_{0\le t\le T}|S_{t-F_\mu^{-1}(u)} - S_{t-F_\nu^{-1}(u)}|\right] du \\
&\le C\int_0^1 \varepsilon\left(|F_\mu^{-1}(u)\wedge T - F_\nu^{-1}(u)\wedge T|\right) du \text{ by (A2)} \\
&\le C\varepsilon\left(\int_0^1 |F_\mu^{-1}(u)\wedge T - F_\nu^{-1}(u)\wedge T|du\right),
\end{aligned}$$

where the last inequality follows from the concavity of $\varepsilon(h)$. The expression inside the brackets is the Wasserstein distance between the measures μ and ν truncated at T. Therefore, from the Kantorovich-Rubinstein theorem, we deduce

$$\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] \le C\varepsilon\left(\int_0^T |F_\mu(t) - F_\nu(t)|dt\right).$$

Step 2. Introduce $\tilde\mu = \mu 1_{[0,T]}$ and $\tilde\nu = \nu 1_{[0,T]} + (\mu([0,T]) - \nu([0,T]))\delta_{2T}$, where δ_{2T} is the point mass at the point $2T$. Then,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] \le C|\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)| + \mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^\infty S_{t-u}\tilde\mu(du) - \int_0^\infty S_{t-u}\tilde\nu(du)\right|\right] \\
&\le C|\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)| \\
&+ (\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+))\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^\infty S_{t-u}\frac{\tilde\mu^+(du) + \tilde\nu^-(du)}{\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+)} - \int_0^\infty S_{t-u}\frac{\tilde\mu^-(du) + \tilde\nu^+(du)}{\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+)}\right|\right]
\end{aligned}$$

Since $\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+) = \tilde\mu^-(\mathbb{R}_+) + \tilde\nu^+(\mathbb{R}_+)$, both measures under the integral sign are probability measures, and we can apply Step 1, which gives:

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}|M_t - N_t|\right] &\le C|\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)| \\
&+ C(\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+))\varepsilon\left(\frac{1}{\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+)}\int_0^T |F_{\tilde\mu^+ + \tilde\nu^-}(t) - F_{\tilde\mu^- + \tilde\nu^+}(t)|dt\right) \\
&= C|\mu(\mathbb{R}_+) - \nu(\mathbb{R}_+)| + C(\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+))\varepsilon\left(\frac{1}{\tilde\mu^+(\mathbb{R}_+) + \tilde\nu^-(\mathbb{R}_+)}\int_0^T |F_\mu(t) - F_\nu(t)|dt\right),
\end{aligned}$$

because $\tilde\mu$ coincides with μ and $\tilde\nu$ coincides with ν on $[0,T]$. Using the properties of the function ε and the definition of $\tilde\mu$ and $\tilde\nu$, we then get (2.5) with a different constant C. □

2.2.2 Laguerre approximation for moving average processes

The aim of this paragraph is to provide heuristic arguments which lead to Laguerre approximation of the moving average. A rigorous justification with convergence rate is given in Proposition

2.2.2.

We would like to find a finite-dimensional approximation to M in (2.4), that is, find n processes $Y^1, \ldots, Y^n$ such that $(S, Y^1, \ldots, Y^n)$ are jointly Markov, and M_t is approximated in some sense to be made precise later by M_t^n which depends deterministically on $S_t, Y_t^1, \ldots, Y_t^n$.

Since M in (2.4) is linear in S, it is natural to require that the approximation also be linear. Therefore, we assume that $Y = (Y^1, \ldots, Y^n)$ satisfies the linear SDE

$$dY_t = -AY dt + \mathbf{1}(\alpha S_t dt + \beta dS_t), \tag{2.6}$$

where A is an $n \times n$ matrix, $\mathbf{1}$ is a n-dimensional vector with all components equal to 1 and α and β are constants. Similarly, the approximation is given by a linear combination of the components of Y: $M^n = B^\perp Y$, where B is a vector of size n and $\perp$ denotes the matrix transposition.

The solution to (2.6) can be written as

$$Y_t = e^{-At} Y_0 + \int_0^t e^{-A(t-s)} \mathbf{1}(\alpha S_s ds + \beta dS_s)$$

or, assuming stationarity, as

$$Y_t = \int_{-\infty}^t e^{-A(t-s)} \mathbf{1}(\alpha S_s ds + \beta dS_s) \quad \text{and} \quad M_t^n = \int_{-\infty}^t B^\perp e^{-A(t-s)} \mathbf{1}(\alpha S_s ds + \beta dS_s).$$

Integration by parts then yields:

$$M_t^n = \beta B^\perp \mathbf{1} S_t + \int_{-\infty}^t B^\perp (\alpha - A\beta) e^{-A(t-s)} \mathbf{1} S_s ds := K_n S_t + \int_{-\infty}^t h_n(t-u) S_u du.$$

Recalling the structure of the matrix exponential, it follows that the function h_n is of the form

$$h_n(t) = \sum_{k=1}^K e^{-p_k t} \sum_{i=0}^{n_k} c_i^k t^i, \tag{2.7}$$

where $n_1 + \ldots + n_K + K = n$ (K is the number of Jordan blocks of A).

Therefore, the problem of finding a finite-dimensional approximation for M boils down to finding an approximation of the form $K_n \delta_0(dt) + h_n(t)dt$ for the measure μ. This problem is well known in signal processing, where the density h of μ is called impulse response function of a system, and h_n is called Hankel approximation of h. For arbitrary μ and n, Hankel approximations may be very hard to find, and in this paper we shall focus on a subclass for which $K = 1$, that is, the function h_n is of the form

$$h_n(t) = e^{-pt} \sum_{i=0}^{n-1} c_i t^i. \tag{2.8}$$

This is known as *Laguerre approximation*, because for a fixed p, the first n scaled Laguerre functions (defined below) form an orthonormal basis of the space of all functions of the form (2.8) endowed with the scalar product of $\mathrm{L}^2([0,\infty))$. See Mäkilä and Wahlberg [90] for a discussion of optimality of Laguerre approximations among all approximations of type (2.7).

Definition 2.2.1. Fix a scale parameter $p > 0$. The scaled Laguerre functions $(L_k^p)_{k\geq 0}$ are defined on $[0,\infty)$ by:

$$L_k^p(t) = \sqrt{2p}\, P_k(2pt)e^{-pt}, \quad \forall k \geq 0 \tag{2.9}$$

in which $(P_k)_{k\geq 0}$ is the family of Laguerre polynomials explicitly defined on $[0,+\infty)$ by:

$$P_k(t) = \sum_{i=0}^{k} \binom{k}{k-i} \frac{(-t)^i}{i!}, \quad \forall k \geq 0 \tag{2.10}$$

or recursively by:

$$\begin{cases} P_0(t) = 1 \\ P_1(t) = 1 - t \\ P_{k+1}(t) = \frac{1}{k+1}\left((2k+1-t)P_k(t) - kP_{k-1}(t)\right), \forall k \geq 1. \end{cases} \tag{2.11}$$

The scaled Laguerre functions $(L_k^p)_{k\geq 0}$ form an orthonormal basis of the Hilbert space L^2:

$$\forall (j,k), \quad \langle L_j^p, L_k^p \rangle = \delta_{j,k}.$$

We referred in Appendix Section 2.A some useful properties of Laguerre polynomials $(P_k)_{k\geq 0}$.

Fix now an order $n \geq 1$ of truncation of the series and $p > 0$ a scale parameter. In view of Lemma 2.2.1, we propose the following Laguerre approximation of the moving average process M in (2.4):

- Let $H(x) = \mu([x,+\infty))$.
- Compute the Laguerre coefficients of the function H:

$$A_k^p = \langle H, L_k^p \rangle, \quad \forall k = 0, \ldots, n-1$$

  Set:

$$H_n^p(t) = \sum_{k=0}^{n-1} A_k^p L_k^p(t) \quad \text{and} \quad h_n^p(t) = -\frac{d}{dt} H_n^p(t) \tag{2.12}$$

- Approximate the moving average M with

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \int_0^\infty h_n^p(u) S_{t-u} du, \quad \forall t \geq 0. \tag{2.13}$$

The approximation proposed in (2.13) (and in particular the correction coefficient in front of S_t) is chosen so that the total mass of the weighting measure of the approximate moving average $M_t^{n,p}$, namely $(H(0) - H_n^p(0))\delta_0(dx) + h_n^p(x)dx$, is equal to the total mass of the weighting measure μ of the exact moving average. In particular, such an approximation becomes *exact* for a constant asset price S:

$$S_t = S_0, \forall t \geq 0 \Longrightarrow M_t^{n,p} = S_0 \left[H(0) - H_n^p(0) + \int_0^\infty h_n^p(u)du \right] = S_0 \int_0^\infty \mu(du) = M_t, \forall t \geq 0.$$

As a consequence, in view of Lemma 2.2.1, the distance between M in (2.4) and $M^{n,p}$ in (2.13):

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t - M_t^{n,p}| \right]$$

will be controlled by the L^2-distance between H and its Laguerre expansion based approximation H_n^p in (2.12). Its convergence to 0 as n goes to infinity will be ensured by the L^2-basis property of the scaled Laguerre functions, see hereafter in Proposition 2.2.2.

Lemma 2.2.2. The function h_n^p in (2.12) can be written as

$$h_n^p = \sum_{k=0}^{n-1} a_k^p L_k^p, \qquad a_k^p = pA_k^p + 2p \sum_{i=k+1}^{n-1} A_i^p. \tag{2.14}$$

Proof. In view of (2.31) from Lemma 2.A.2,

$$h_n^p(t) = \sum_{k=0}^{n-1} A_k^p \left(2p \sum_{i=0}^{k-1} L_i^p(t) + pL_k^p(t) \right) = p \sum_{k=0}^{n-1} A_k^p L_k^p(t) + 2p \sum_{k=0}^{n-1} \left(\sum_{i=k+1}^{n-1} A_i^p \right) L_k^p(t).$$

□

From definitions (2.13) and (2.14), it seems natural to introduce n random processes:

$$\left(X^{p,0}, X^{p,1}, \ldots, X^{p,n-1}\right)$$

defined by:

$$X_t^{p,k} = \int_0^\infty L_k^p(u) S_{t-u} du, \quad \forall t \geq 0, \forall k = 0, \ldots, n-1. \tag{2.15}$$

They will be called the n *Laguerre processes* associated to our approximation and are related to the moving average approximation in (2.13) by:

$$M_t^{n,p} = (H(0) - H_n^p(0))S_t + \sum_{k=0}^{n-1} a_k^p X_t^{p,k}, \quad \forall t \geq 0. \tag{2.16}$$

Proposition 2.2.1. *Let $n \geq 1$ and $p > 0$. The $(n+1)$-dimensional process:*

$$(S, X^{p,0}, X^{p,1}, \ldots, X^{p,n-1})$$

is Markovian. In addition, the initial value of the Laguerre processes are

$$X_0^{p,k} = S_0 (-1)^k \frac{\sqrt{2p}}{p}, \forall k \geq 0. \tag{2.17}$$

Proof. Let us show that:

$$\begin{cases} dX_t^{p,0} = \left(\sqrt{2p} S_t - pX_t^{p,0} \right) dt \\ dX_t^{p,1} = \left(\sqrt{2p} S_t - p \left(2X_t^{p,0} + X_t^{p,1} \right) \right) dt \\ \vdots \\ dX_t^{p,n-1} = \left(\sqrt{2p} S_t - p \left(2 \sum_{k=0}^{n-2} X_t^{p,k} + X_t^{p,n-1} \right) \right) dt. \end{cases}$$

For sake of clarity, we denote here by $X^k \equiv X^{p,k}$. Rewrite (2.15) as:

$$X_t^k = \int_{-\infty}^t L_k^p(t-u)S_u du, \forall k \geq 0.$$

With definition (2.9), as $P_0 = 1$:

$$\begin{aligned} dX_t^0 &= \left(\int_{-\infty}^t \partial_t L_0^p(t-u)S_u du + L_0^p(0)S_t\right) dt \\ &= \left(-pX_t^0 + \sqrt{2p}S_t\right) dt. \end{aligned}$$

Fix now $k \geq 1$. As $P_k(0) = 1$, we get by definition (2.9), properties 2.A.1-(i) and 2.A.1-(ii):

$$\begin{aligned} dX_t^k &= \left(\int_{-\infty}^t \partial_t L_k^p(t-u)S_u du + L_k^p(0)S_t\right) dt \\ &= \left(\int_{-\infty}^t \sqrt{2p}\big[2pP_k'(2p(t-u)) - pP_k(2p(t-u))\big]e^{-p(t-u)}S_u du + \sqrt{2p}S_t\right) dt \\ &= \left(\int_{-\infty}^t \sqrt{2p}\big[-2p\textstyle\sum_{i=0}^{k-1} P_i(2p(t-u)) - pP_k(2p(t-u))\big]e^{-p(t-u)}S_u du + \sqrt{2p}S_t\right) dt \\ &= \left(-2p\textstyle\sum_{i=0}^{k-1} X_t^i - pX_t^k + \sqrt{2p}S_t\right) dt. \end{aligned}$$

(2.17) is straightforward by convention (2.3), definition (2.10) and:

$$\int_0^\infty L_k^p(u)du = \sqrt{2p}\sum_{i=0}^k \binom{k}{k-i}\frac{(-2)^i}{i!}\frac{1}{p}\int_0^\infty y^i e^{-y}dy = \frac{\sqrt{2p}}{p}\sum_{i=0}^k \binom{k}{i}(-2)^i = \frac{\sqrt{2p}}{p}(-1)^k.$$

□

2.2.3 Convergence of the Laguerre approximation

The following Proposition 2.2.2 provides a convergence rate on n of the error of approximation by $M^{n,p}$ of the exact moving average M. As a by-product, see Corollary 2.2.1, we get the convergence to 0 of the pricing error introduced when replacing the moving average M by its approximation $M^{n,p}$ in an American option pricing problem.

Proposition 2.2.2. *Let assumption* (A2) *be satisfied, and suppose that the moving average process M is of the form*

$$M_t = K_0 S_t + \int_0^\infty S_{t-u}h(u)du \tag{2.18}$$

where K_0 is a constant and the function h has compact support, finite variation on $\mathbb{R}$, is constant in the neighborhood of zero and is not a.e. negative on $[0,T]$. Recall the definition of $M^{n,p}$ in (2.13)*. Then:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|M_t - M_t^{n,p}|\right] \leq C\varepsilon(n^{-\frac{3}{4}}).$$

where $C > 0$ is a constant independent of n.

Proof. We shall apply Lemma 2.2.1 to M and $M^{n,p}$. The measures μ and ν are defined by $\mu(dx) = K_0\delta_0(dx) + h(x)dx$ and $\nu(dx) = (H(0) - H_n^p(0))\delta_0(dx) + h_n^p(x)dx$. As already mentioned, these measures have the same mass, and the first term in estimate (2.5) disappears. In addition,

$$|\mu([0,T]) - \nu([0,T])| = |H(T) - H_n^p(T)|,$$

which remains bounded by Lemma 2.A.4. Let us show that $\nu^-([0,T])$ is bounded as well. For this it is enough to prove that $\|(H_n^p)'\|_2$ is bounded on n. Set $c_k^p := \langle h, L_k^p \rangle$, the Laguerre coefficients of h. An integration by parts followed by the use of (2.31) in Lemma 2.A.2 shows that

$$\begin{aligned} c_k^p &= L_k^p(0)H(0) + \int_0^\infty H(t)\left(L_k^p(t)\right)' dt \\ &= \sqrt{2p}H(0) - 2p\sum_{i=0}^{k-1} A_i^p - pA_k^p, \quad \forall k \geq 0. \end{aligned} \tag{2.19}$$

By definition of H_n^p and expression for a_k^p in (2.14), this leads to:

$$a_k^p = c_k^p - \sqrt{2p}\left[H(0) - H_n^p(0)\right]. \tag{2.20}$$

We have thus:

$$\|(H_n^p)'\|_2^2 = \sum_{k\leq n-1} |a_k^p|^2 \leq 2\sum_{k\leq n-1} \left|c_k^p\right|^2 + 2\sum_{k\leq n-1} \left|\sqrt{2p}\left[H(0) - H_n^p(0)\right]\right|^2 = \mathcal{O}(n^{-\frac{1}{2}})$$

by Lemma 2.A.4 and using $\sqrt{2p}\left[H(0) - H_n^p(0)\right] = c_n^p + pA_n^p$ issued from (2.19). Therefore, there exists a constant $C < \infty$, which does not depend on n, such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t - M_t^{n,p}|\right] \leq C\varepsilon\left(\frac{1}{\int_0^T h^+(t)dt}\int_0^T |H(t) - H_n^p(t)|dt\right).$$

By Cauchy-Schwartz inequality and Lemma 2.A.4,

$$\int_0^T |H(t) - H_n^p(t)|dt \leq \sqrt{T}\|H - H_n^p\|_2 = \sqrt{T}\left(\sum_{k\geq n}\left|A_k^p\right|^2\right)^{\frac{1}{2}} = \mathcal{O}(n^{-\frac{3}{4}}),$$

from which the result follows using the properties of ε and the fact that h is not a.e. negative on $[0,T]$ (which means that $\int_0^T h^+(t)dt > 0$). □

Propositions 2.2.1 and 2.2.2 allow us to introduce a *finite-dimensional approximation* to infinite-dimensional moving average option problem. One can indeed approximate the price of the American option whose payoff depends on the moving average M:

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] \tag{2.21}$$

by the solution to the $(n+1)$-finite-dimensional problem:

$$\sup_{\tau\in\mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \tag{2.22}$$

in which:

- $n \geq 1$ is the order of the approximation (number of scaled Laguerre functions introduced),
- $p > 0$ is the scale parameter of the Laguerre functions, which can be chosen to minimize the L^2-distance between H and H_n^p.

Remark 2.2.1. In the framework of the multi-assets model introduced in Remark 2.1.1, the same kind of approximation based on a finite Laguerre expansion can be formulated. The resulting dimension of the approximate problem is $(d+n)$. In addition, the same estimate of the pricing error as in Corollary 2.2.1 is obtained as soon as:

- Assumption (A1) is replaced by the Lipschitz continuity of ϕ is its $(d+1)$-th variable.
- (A2) is satisfied by $S^j, \forall j \in \mathcal{J}$.

Remark 2.2.2 (Application to Swing options)**.** Our method can be clearly used for pricing American-style options with a multiple-exercise feature. The Laguerre approximation of moving average processes remains the same and so does the bound on the pricing error as function of n.

Corollary 2.2.1. *Let assumptions* (A1) *and* (A2) *be satisfied, and assume that the moving average process M satisfies the assumptions of Proposition* (2.2.1)*. Then the pricing error admits the bound*

$$\mathcal{E}_{\text{pricing}}(n,p) := \left| \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] - \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \right| \leq C\varepsilon(n^{-\frac{3}{4}}).$$

where $C > 0$ is a constant independent of n.

Proof. We have first:

$$\begin{aligned}
\forall \tau, \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] &= \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] + \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right) - \phi\left(S_\tau, M_\tau^{n,p}\right)\right] \\
\Longrightarrow \sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] &= \sup_\tau \Big(\mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] + \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right) - \phi\left(S_\tau, M_\tau^{n,p}\right)\right] \Big) \\
&\leq \sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] + \sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right) - \phi\left(S_\tau, M_\tau^{n,p}\right)\right]
\end{aligned}$$

In consequence:

$$\sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] - \sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \leq \sup_\tau \mathbb{E}\left|\phi\left(S_\tau, M_\tau\right) - \phi\left(S_\tau, M_\tau^{n,p}\right)\right|$$

By symetry and (A1), we get:

$$\begin{aligned}
\left| \sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau\right)\right] - \sup_\tau \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right] \right| &\leq C \sup_\tau \mathbb{E}\left|M_\tau - M_\tau^{n,p}\right| \\
&\leq C\, \mathbb{E}\left[\sup_{0 \leq t \leq T} \left|M_t - M_t^{n,p}\right| \right],
\end{aligned}$$

and the result follows from Proposition 2.2.2. □

Remark 2.2.3. A faster convergence rate might probably be obtained when scaling the Laguerre functions optimally with respect to the L^2-distance between H and H_n^p. We refer to Section 3.1 for such a (numerical) improvement in the framework of uniformly-weighted moving averages.

2.3 Uniformly-weighted moving averages

Going back to the problem formulated in Section 2.1, the moving average process X in (2.1) admits an uniform weighting measure

$$\mu(dx) = \frac{1}{\delta}\mathbb{1}_{[l,l+\delta]}(x)dx,$$

which satisfies the assumptions of Proposition 2.2.2. In the sequel, we shall denote:

$$h := h(\delta, l) = \frac{1}{\delta}\mathbb{1}_{[l,l+\delta]}. \tag{2.23}$$

The general methodology introduced in Section 2.2 is applicable. In present case:

$$H(x) := H(\delta, l)(x) = \frac{1}{\delta}\left\{(\delta + l - x)^+ - (l - x)^+\right\}. \tag{2.24}$$

In the rest of this Part (in particular in following Chapter 3), for ease of notation and when it is not confusing, we might avoid the dependance in (δ, l) in the notations for h, H, their corresponding Laguerre coefficients $c_k^p = \langle h, L_k^p\rangle$, $A_k^p = \langle H, L_k^p\rangle$, for H_n^p defined in (2.12) and a_k^p, recall (2.20):

$$a_k^p \;=\; c_k^p - \sqrt{2p}\,[1 - H_n^p(0)]\,. \tag{2.25}$$

For purpose of simplification in computation of the numerical method (see Section 3.1 in the following chapter), we give in Proposition 2.3.1 more tractable expressions for coefficients A_k^p and c_k^p, which allow an *exact computation*. Besides, we refer in Proposition 2.3.2 the useful scaling properties verified by c_k^p, A_k^p and a_k^p.

Proposition 2.3.1. *Let $\delta > 0$, $l \geq 0$ and $p > 0$.*

(i) The Laguerre coefficients of H in (2.24) are related to the Laguerre coefficients of h in (2.23) via

$$A_k^p = (-1)^k\frac{\sqrt{2p}}{p} - \frac{1}{p}c_k^p - \frac{2}{p}\sum_{i=0}^{k-1}(-1)^{k-i}c_i^p, \quad \forall k \geq 0. \tag{2.26}$$

(ii) The Laguerre coefficients of h can be computed from the values of Laguerre polynomials:

$$\begin{aligned} c_k^p(\delta, l) := c_k^p = \frac{\sqrt{2p}}{\delta p}\Big[&\left(e^{-pl}P_k(2pl) - e^{-p(l+\delta)}P_k(2p(l+\delta))\right)\\ &+2\sum_{i=1}^{k}(-1)^i\left(e^{-pl}P_{k-i}(2pl) - e^{-p(l+\delta)}P_{k-i}(2p(l+\delta))\right)\Big] \end{aligned} \tag{2.27}$$

or explicitly as

$$c_k^p(\delta, l) := c_k^p = \sqrt{2p}\sum_{i=0}^{k}\binom{k}{k-i}(-2)^i C_i, \quad \forall k \geq 0, \tag{2.28}$$

where $\forall i \geq 0$:

$$C_i = \frac{e^{-lp}}{\delta p}\left[\left(1 - e^{-\delta p}\right) + \sum_{j=1}^{i}\frac{p^j}{j!}\left(l^j - (l+\delta)^j e^{-\delta p}\right)\right].$$

Proof. *(i)* Write first:

$$A_k^p = \int_0^l L_k^p(x)dx + \frac{l+\delta}{\delta}\int_l^{l+\delta} L_k^p(x)dx - \frac{1}{\delta}\int_l^{l+\delta} xL_k^p(x)dx.$$

By Lemma 2.A.2, we have following expression for the integral of the (scaled) Laguerre functions:

$$\mathcal{L}_k^p(x) := \int_x^\infty L_k^p(t)dt = \frac{1}{p}L_k^p(x) + \frac{2}{p}\sum_{i=0}^{k-1}(-1)^{k-i}L_i^p(x), \quad \forall k \geq 0. \tag{2.29}$$

As $-\mathcal{L}_k^p(x)$ is a primitive of $L_k^p(x)$, by integration by parts:

$$\int_0^\delta xL_k^p(x)dx = l\mathcal{L}_k^p(l) - (l+\delta)\mathcal{L}_k^p(l+\delta) + \int_l^{l+\delta}\mathcal{L}_k^p(x)dx$$

leading by (2.29) to:

$$\begin{aligned} A_k^p &= \int_0^\infty L_k^p(x)dx - \frac{1}{\delta p}\int_l^{l+\delta} L_k^p(x)dx - \frac{2}{\delta p}\sum_{i=0}^{k-1}(-1)^{k-i}\int_l^{l+\delta} L_i^p(x)dx \\ &= (-1)^k\frac{\sqrt{2p}}{p} - \frac{1}{p}c_k^p - \frac{2}{p}\sum_{i=0}^{k-1}(-1)^{k-i}c_i^p. \end{aligned}$$

(ii) Set first $l = 0$. Then

$$\begin{aligned} c_k^p(\delta,0) &= \frac{1}{\delta}\left[(-1)^k\frac{\sqrt{2p}}{p} - \frac{1}{p}L_k^p(\delta) + \frac{2}{p}\sum_{i=1}^{k}(-1)^i L_{k-i}^p(\delta)\right] \\ &= \frac{\sqrt{2p}}{\delta p}\left[(1-e^{-p\delta}P_k(2p\delta)) + 2\sum_{i=1}^{k}(-1)^i(1-e^{-p\delta}P_{k-i}(2p\delta))\right], \end{aligned}$$

by (2.29) and (2.9). This leads to the expected expression for $c_k^p(\delta,l)$, $\forall l \geq 0$. (2.28) can be obtained by a straightforward computation using (2.9) and (2.10). Indeed,

$$c_k^p(\delta,l) = \sqrt{2p}\sum_{i=0}^{k}\binom{k}{k-i}\frac{(-2)^i}{i!}\left(\frac{1}{\delta p}\int_{lp}^{(l+\delta)p} y^i e^{-y}dy\right)$$

and a simple integration by parts combined with a recursion on i allows to ensure that the last term in brackets is equal to C_i. □

Proposition 2.3.2. *Let $\delta > 0$, $l \geq 0$ and $p > 0$. For any $k \geq 0$, the coefficients c_k^p, A_k^p and a_k^p satisfy following scaling properties:*

$$\begin{aligned} \forall \lambda > 0, \quad c_k^p(\delta,l) &= c_k^{p/\lambda}(\delta\lambda, l\lambda)\sqrt{\lambda} \\ A_k^p(\delta,l) &= A_k^{p/\lambda}(\delta\lambda, l\lambda)/\sqrt{\lambda} \\ a_k^p(\delta,l) &= a_k^{p/\lambda}(\delta\lambda, l\lambda)\sqrt{\lambda} \end{aligned}$$

Proof. This comes directly from the definitions of these coefficients and the scaling property of the Laguerre functions (2.9). □

2.A Some properties of the Laguerre polynomials

Lemma 2.A.1. The Laguerre polynomials $(P_k)_{k\geq 0}$ belong to $\mathcal{C}^\infty([0,+\infty))$ and:

(i) $\forall k \geq 1, tP_k'(t) - kP_k(t) + kP_{k-1}(t) = 0$

(ii) $\forall k \geq 1, \frac{k}{t}\left(P_k(t) - P_{k-1}(t)\right) = -\sum_{i=0}^{k-1} P_i(t)$

Proof. (i) can be found for example in Szegö [106] and (ii) is a direct consequence of (2.11). □

Lemma 2.A.2. The definite integrals and derivatives of Laguerre functions can be computed using the following formulas:

$$\int_t^\infty e^{-s/2}P_n(s)ds = 2e^{-t/2}P_n(t) + 4e^{-t/2}\sum_{k=1}^{n}(-1)^k P_{n-k}(t). \tag{2.30}$$

$$\left(e^{-t/2}P_n(t)\right)' = -\sum_{k=0}^{n-1} e^{-t/2}P_k(t) - \frac{1}{2}e^{-t/2}P_n(t). \tag{2.31}$$

Proof. This follows, after some computations, from the contour integral representation of Laguerre polynomials:

$$P_n(t) = \frac{1}{2\pi i}\oint \frac{e^{-\frac{ts}{1-s}}}{(1-s)s^{n+1}}ds.$$

□

Lemma 2.A.3. The Laguerre functions and their integrals admit the following representation in terms of Bessel functions:

$$e^{-x/2}P_n(x) = \sum_{k=0}^{\infty} A_k \left(\frac{x}{\nu}\right)^{k/2} J_k(\sqrt{\nu x}), \quad \nu = 4n+2 \tag{2.32}$$

$$I_n(x) := \int_0^x e^{-x'/2}P_n(x')dx' = 2\sum_{k=0}^{\infty} A_k \left(\frac{x}{\nu}\right)^{(k+1)/2} J_{k+1}(\sqrt{\nu x}) \tag{2.33}$$

$$\int_0^x I_n(x')dx' = 4\sum_{k=0}^{\infty} A_k \left(\frac{x}{\nu}\right)^{(k+2)/2} J_{k+2}(\sqrt{\nu x}), \tag{2.34}$$

where $A_0 = 1$, $A_1 = 0$, $A_2 = \frac{1}{2}$ and other A_i-s satisfy the equation $(m+2)A_{m+2} = (m+1)A_m - \frac{\nu}{2}A_{m-1}$. The series converge uniformly in x on any compact interval.

Proof. The first formula is from Erdélyi et al. [50]. The other two follow readily using the integration formula for Bessel functions:

$$\int_0^1 x^{\nu+1}J_\nu(ax)dx = a^{-1}J_{\nu+1}(a).$$

□

Lemma 2.A.4. Let μ be a finite signed measure on $[0,\infty)$, with bounded support which does not contain zero. Let $\{c_n\}$ denote the Laguerre coefficients of the function $h(x) := \mu([x,\infty))$ and $\{A_n\}$ denote the Laguerre coefficients of the function $H(x) := \int_x^\infty h(t)dt$. Then

$$c_n = \mathcal{O}(n^{-3/4}) \quad \text{and} \quad A_n = \mathcal{O}(n^{-5/4}).$$

In addition, for $x > 0$ fixed, $e^{-x/2}P_n(x) = \mathcal{O}(n^{-1/4})$.

Proof. This result follows from Lemma 2.A.3, using the asymptotic expansion for Bessel functions

$$J_n(x) = (\frac{1}{2}\pi x)^{-1/2}\cos\left(x - \frac{\pi}{2}n - \frac{\pi}{4}\right) + \mathcal{O}(x^{-3/2}),$$

which holds uniformly (cf. Erdélyi et al. [50]) on bounded domains outside a neighborhood of zero. □

Chapter 3

Methods for pricing moving average options

Let us consider the problem of pricing moving average options with time delay in a single-asset framework[1]:

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, X_\tau = \frac{1}{\delta}\int_{t-l-\delta}^{t-l} S_u du\right)\right]. \tag{3.1}$$

We propose in Section 3.1 a fully implementable algorithm to compute the solution to the finite-dimensional approximate problem introduced in Chapter 2:

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau, M_\tau^{n,p}\right)\right], \tag{3.2}$$

in which the moving average X is replaced by its Laguerre expansion-based approximate $M^{n,p}$ defined in (2.16). It is a $(n+1)$-dimensional optimal stopping time problem (see Proposition 2.2.1) and the convergence of the approximate option price to the true price is ensured when letting n go to infinity (see Corollary 2.2.1).

In our numerical method, the moving average process approximation is improved by *optimal scaling* of the basis of Laguerre functions. Since the dimension of the problem may be high, we use a Monte Carlo technique. Our numerical approach corresponds to the one from Longstaff and Schwartz [82] and the computation of conditional expectations is done with a regression based approach. In particular, we shall use the improved technique of adaptative local basis proposed by Bouchard and Warin [21]. This will allow us to deal with state vectors up to dimension 8.

In Section 3.2, we introduce a benchmark method for solving problem (3.1) (also based on the Longstaff and Schwartz algorithm). However, this discrete-time solving method becomes computationally unfeasible, for applications in which the averaging window and/or the time lag is large. Indeed, it has a state dimension equal to the number of time step within the averaging window plus the number of time steps within the time lag l. On the contrary, recall that the dimension of the discrete-time version of our approximate problem (3.2) is equal to $(n+1)$ *whatever the values of* δ *and* l. In a second time, we present an other numerical method, based on a common practice (see also Broadie and Cao [22]) when pricing moving average-type American-style options. It is a non Markovian approximate method which consists in computing the conditional expectations implied in the backward induction algorithm using only two explanatory variables: the price value at the considered time and its moving average. Such a method will be used as a lower reference in our numerical experiments.

[1] For the sake of simplicity, we restrict ourselves to uniformly weighted moving averages commonly encountered in practice and set the interest rate to zero without loss of generality.

In the final present, we present two immediate generalizations of our numerical methods, which are useful for pricing oil-indexed gas contracts commonly encountered in the gas market (see Section 4.2 in next Chapter 4). We describe first an extension to the case of multi-assets based moving average options as described in Remark 2.1.1. Finally, we show how to deal with non-continuously updated strike prices X (namely, if it is updated at some fixed dates).

3.1 Laguerre approximation-based numerical method

Main tools of the Laguerre approximation for uniformly-weighted moving averages have been detailed in Section 2.3 of previous Chapter 2. We decompose the numerical method in three steps: whereas the first one consists in optimizing the moving average approximation by optimal scaling of the Laguerre functions, the two following steps corresponds simply to the classical discrete-time forward and backward schemes for solving American-style option problem (3.2).

Step 1: Optimal scaling of the Laguerre basis functions

Fix an order $n \geq 1$ of Laguerre approximation, corresponding to the number of Laguerre basis functions used in the finite expansion H_n^p for approximating H defined in (2.24). The first step is to determine the optimal scale parameter $p_{\text{opt}}(n, \delta, l)$ of the sequence of Laguerre functions $\{L_0^p, \ldots, L_{n-1}^p\}$. Indeed, this choice gives the best convergence rate on n of the L^2-distance between H and H_n^p.

Proposition 3.1.1. *Let $\delta > 0$ and $l \geq 0$. The optimal scale parameter $p_{\text{opt}}(n, \delta, l)$ of the truncated basis of Laguerre functions $\{L_0^p, \ldots, L_{n-1}^p\}$ solution to:*

$$p_{\text{opt}}(n, \delta, l) = \arg\min_{p>0} \|H - H_n^p\|_2^2 \tag{3.3}$$

satisfies the scaling property:

$$\forall \lambda > 0, \quad p_{\text{opt}}(n, \delta, l) = \frac{p_{\text{opt}}(\delta/\lambda, l/\lambda, n)}{\lambda}.$$

The resulting L^2-distance is scale invariant, namely:

$$\forall \lambda > 0, \quad \left\| \left[H - H_n^{p_{\text{opt}}(n,\delta,l)} \right](\delta, l) \right\|_2 = \left\| \left[H - H_n^{\frac{p_{\text{opt}}(\delta/\lambda, l/\lambda, n)}{\lambda}} \right](\delta/\lambda, l/\lambda) \right\|_2 .$$

Proof. This comes from the definition of the L^2-distance between H and H_n^p:

$$\|H - H_n^p\|_2^2 = \left(\frac{\delta}{3} + l \right) - \sum_{k=0}^{n-1} \left| A_k^p \right|^2 \tag{3.4}$$

and the scaling property of the coefficients A_k^p in Proposition 2.3.2. □

The minimization problem (3.3) admits an unique solution. Finding an explicit formula to $p_{\text{opt}}(n, \delta, l)$ does not seem to be possible, but finding a numerical solution is easy using the explicit expressions (2.26), (2.28), and (3.4).

► Let us first consider the case without time lag ($l = 0$). Once p_{opt} is computed for $(n, 1, 0)$, Proposition 3.1.1 gives the value of $p_{\text{opt}}(n, \delta, 0)$, for any $\delta > 0$:

$$p_{\text{opt}}(n, \delta, 0) = \frac{p_{\text{opt}}(n, 1, 0)}{\delta}.$$

It means that you actually need to compute only once a set of values $p_{\text{opt}}(1, 0, n)$ for reasonable n and keep them for example in a table for further use in Step 2 and Step 3. Table 3.1 gives the values for $p_{\text{opt}}(n, 1, 0)$ for the first 10 values of n computed with an accuracy of 10^{-3}.

n	1	2	3	4	5	6	7	8	9	10
$p_{\text{opt}}(n, 1, 0)$	2.149	4.072	6.002	4.234	5.828	7.473	9.155	10.866	9.153	10.726

Table 3.1: Optimal scale parameters for approximating $H(1, 0)(x) = (1 - x)^+$.

Figure 3.1 illustrates the approximation of H by the truncated Laguerre expansion $H_n^{p_{\text{opt}}(n)}$ for $n = 1, 3, 7$ Laguerre basis functions (with $\delta = 1$).

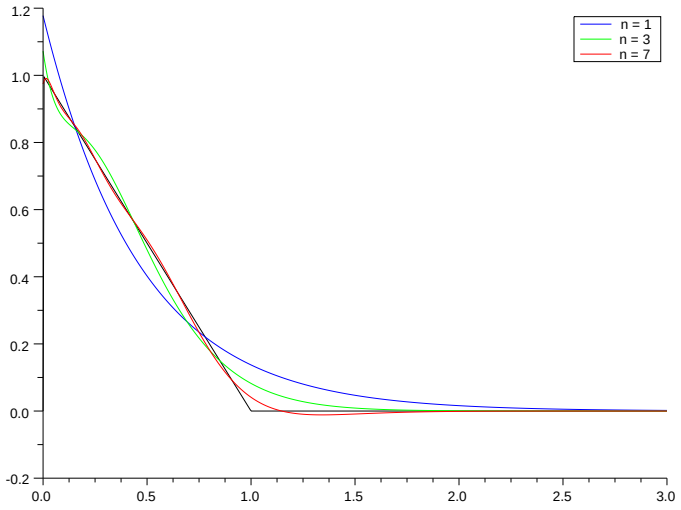

Figure 3.1: Laguerre approximation of the function $H(1, 0)$.

Figure 3.2 shows the corresponding error $||H - H_n^{p_{\text{opt}}(n)}||_2$ as a function of n. The error is less than 5% already with $n = 3$ Laguerre functions. A simple least squares estimation by a power function gives a behavior in $\mathcal{O}(n^{-1.06})$. Besides, the scale invariance of L^2-error between H and H_n^p (see Proposition 3.1.1) implies that the behaviour shown in Figure 3.2 holds for any $\delta > 0$. The optimal scaling of the Laguerre functions leads thus to a *convergence rate improvement*, in comparison to the convergence rate in $\mathcal{O}(n^{-\frac{3}{4}})$ (uniform in p) given by Lemma 2.A.4 for error $\|H - H_n^p\|_2$, recall (3.4).

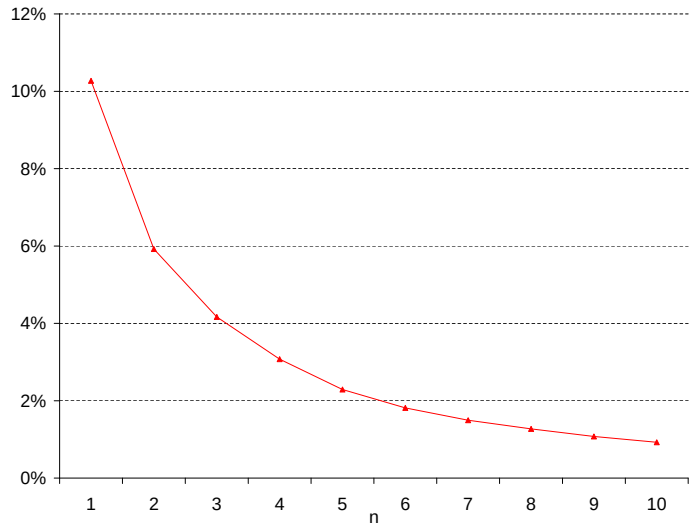

Figure 3.2: L^2-error of the approximation as function of the number of Laguerre functions.

▶ When $l > 0$, Proposition 3.1.1 implies in the same way:

$$p_{\text{opt}}(n, \delta, l) = \frac{p_{\text{opt}}(n, 1, l/\delta)}{\delta}, \quad \forall \delta > 0, \forall l \geq 0.$$

Table 3.2 gives the values for $p_{\text{opt}}(n, 1, \alpha), \alpha \in \left\{\frac{1}{2}, 1, 2\right\}$ for the first 10 values of n computed with an accuracy of 10^{-3}.

n	1	2	3	4	5	6	7	8	9	10
$p_{\text{opt}}(n, 1, \frac{1}{2})$	1.229	2.700	4.138	2.973	3.999	3.206	5.943	7.189	8.488	7.277
$p_{\text{opt}}(n, 1, 1)$	0.831	1.861	2.939	4.020	3.005	3.909	4.761	3.964	3.415	3.941
$p_{\text{opt}}(n, 1, 2)$	0.501	1.131	1.800	2.488	1.846	2.442	3.055	3.678	4.307	3.699

Table 3.2: Optimal scale parameters for approximating $H(1, \alpha)$ with $\alpha \in \left\{\frac{1}{2}, 1, 2\right\}$.

We report in Figures 3.3 and 3.4, the approximation of H by truncated Laguerre expansions $H_n^{p_{\text{opt}}(n)}$ when $(\delta, l) = (1, 1)$, $(1, \frac{1}{2})$ and $(\delta, l) = (1, 2)$ respectively.

Figure 3.5 shows the corresponding (relative) L^2-errors made as functions of n. In comparison to previous case (when $l = 0$), it appears that the number of Laguerre basis functions necessary to approximate H is bigger for an equivalent accuracy of the approximation. This number increases when δ is small with respect to l: the error is less than 5% from $n = 5$ in the two first cases whereas at least $n = 7$ Laguerre functions are required when $\delta = l/2$. This is due to the fact that the density of the weighting measure of the moving average has *two points of discontinuity* when $l > 0$ against only one for the Heaviside function implied in the case when $l = 0$. In addition, as shown in Figure 3.4, the Laguerre basis has more difficulty to capture these two discontinuity points when they become closer (corresponding to $\delta \ll l$).

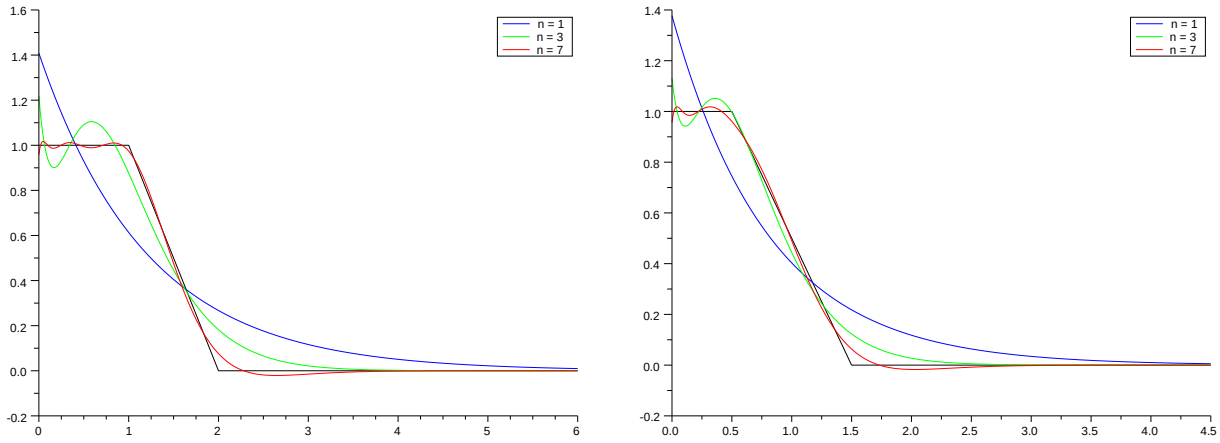

Figure 3.3: Laguerre approximation of $H(\delta, l)$ with $\delta = 1$ and $l = 1$ (left), $l = \frac{1}{2}$ (right).

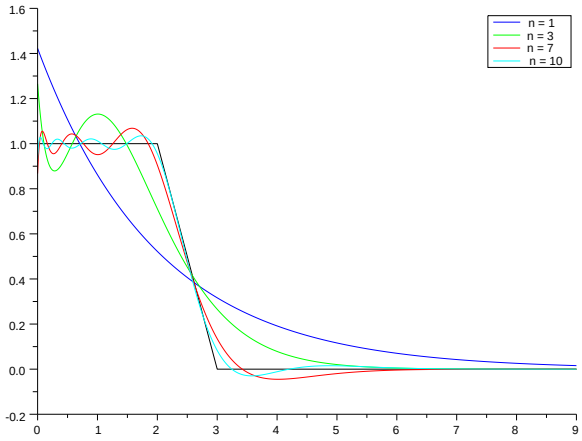

Figure 3.4: Laguerre approximation of the function $H(\delta, l)$ with $\delta = 1$ and $l = 2$.

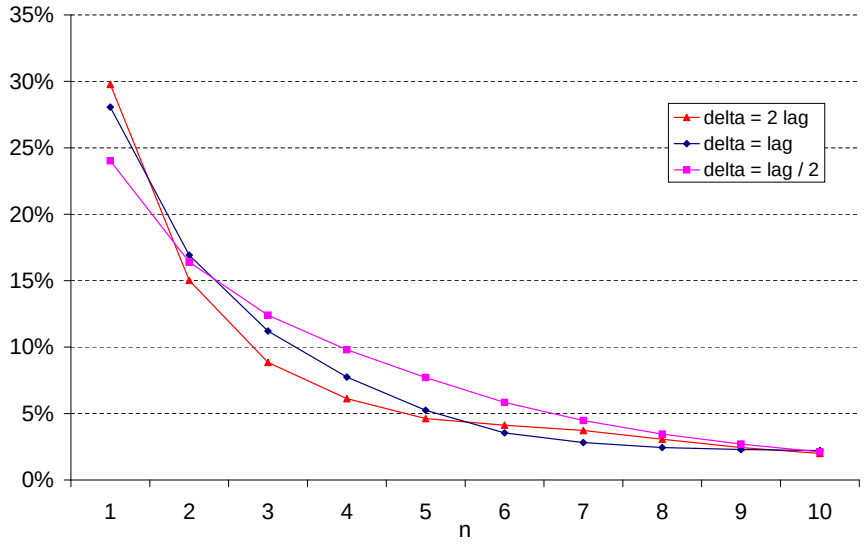

Figure 3.5: L^2-error of the approximation ($l > 0$).

Step 2: Forward simulation in discrete time

The two following steps correspond to the computation in a discrete-time setting of the price of the American option (3.2). Fix an order $n \geq 1$ of Laguerre approximation. In the sequel, for ease of notation, we shall denote the optimal scale parameter $p_{\text{opt}} := p_{\text{opt}}(n, \delta, l)$. Recall that the $(n+1)$-dimensional Markovian state of this problem is:

$$\left(S, X^{p_{\text{opt}},0}, X^{p_{\text{opt}},1}, \ldots, X^{p_{\text{opt}},n-1}\right)$$

in which the $(X^{p_{\text{opt}},k})_{k\geq 0}$ are the Laguerre processes defined in (2.15). We consider a numerical computation on a regular discrete-time grid with $N \geq 1$ time steps with size $\Delta t = \frac{T}{N}$:

$$\pi = \{t_0 = 0, t_1, \ldots, t_N = T\}.$$

Let $N_\delta \in \mathbb{N}^*$ and $N_l \in \mathbb{N}$ be respectively the number of time steps within the averaging window with length δ and the time lag l:

$$N_\delta = \frac{\delta}{\Delta t} = \frac{\delta}{T}N, \quad N_l = \frac{l}{\Delta t} = \frac{l}{T}N.$$

We assume that the spot price S can be simulated on π either perfectly or using an Euler scheme, denote its discrete-time version by

$$\left\{S^\pi_{t_0} = S_0, S^\pi_{t_1}, \ldots, S^\pi_{t_N}\right\},$$

extend this definition on $[0, T]$ by

$$S^\pi_t = S^\pi_{t_{i+1}}, \forall t \in (t_i, t_{i+1}] \tag{3.5}$$

and shall also apply convention (2.3) to S^π. We define the discrete-time version X^π of the moving average process X by

$$X^\pi_{t_i} = \frac{1}{N_\delta} \sum_{j=i-N_l-N_\delta+1}^{i-N_l} S^\pi_{t_j}, \quad \forall t_i \in \pi, t_i \geq \delta + l \tag{3.6}$$

which corresponds to a simple arithmetic average. This definition is consistent with our definition of $(S^\pi_t)_{0\leq t\leq T}$ in (3.5) as

$$X^\pi_{t_i} = \frac{1}{\delta} \int_{t_i-l-\delta}^{t_i-l} S^\pi_t dt.$$

An advantage of our Laguerre functions-based method is that it allows to *exactly compute* the Laguerre processes $(X^{p,k})_{k\geq 0}$ on any time grid, as soon as a discrete-time version of the price process S is available.

Proposition 3.1.2. *Let $p > 0$ and $k \geq 0$. The discrete-time version $X^{p,k,\pi}$ of the Laguerre process $X^{p,k}$ defined in (2.15) on time grid π is:*

$$X^{p,k,\pi}_{t_i} = \sum_{j=1}^{i} \left(S^\pi_{t_j} - S^\pi_{t_{j-1}}\right) (i-j+1)\Delta t \; c^p_k((i-j+1)\Delta t, 0) + S_0(-1)^k \frac{\sqrt{2p}}{p}, \forall t_i \in \pi$$

in which $c^p_k(\delta, 0) = \langle \frac{1}{\delta}\mathbb{1}_{[0,\delta]}, L^p_k\rangle_2$ have the explicit expression given in (2.28).

Proof. Let $k \geq 0$. Proposition 2.2.1 gives $X^{p,k,\pi}_{t_0} = S_0(-1)^k \frac{\sqrt{2p}}{p}$. Let $t_i \in \pi, t_i > t_0$. The computation of $X^{p,k,\pi}_{t_i}$ is straightforward by its definition in (2.15):

$$X^{p,k,\pi}_{t_i} = \int_0^\infty L^p_k(v) S^\pi_{t_i-v} dv.$$

Together with (2.3) and (3.5), we get

$$\begin{aligned}
X^{p,k,\pi}_{t_i} &= \int_0^\infty L^p_k(u) S^\pi_{t_i-u} du \\
&= \left(\int_0^{t_i-t_{i-1}} + \int_{t_i-t_{i-1}}^{t_i-t_{i-2}} + \ldots + \int_{t_i-t_2}^{t_i-t_1} + \int_{t_i-t_1}^{t_i}\right) L^p_k(v) S^\pi_{t_i-v} dv + \int_{t_i}^\infty L^p_k(v) S^\pi_{t_i-v} dv \\
&= S^\pi_{t_i} \Delta t \; c^p_k(\Delta t, 0) + S^\pi_{t_{i-1}} \left[2\Delta t \; c^p_k(2\Delta t, 0) - \Delta t \; c^p_k(\Delta t, 0)\right] + \ldots \\
&\quad + S^\pi_{t_1} \left[i\Delta t \; c^p_k(i\Delta t, 0) - (i-1)\Delta t \; c^p_k((i-1)\Delta t, 0)\right] + S^\pi_0 \left[(-1)^k \frac{\sqrt{2p}}{p} - i\Delta t \; c^p_k(i\Delta t, 0)\right] \\
&= \left(S^\pi_{t_i} - S^\pi_{t_{i-1}}\right) \Delta t \; c^p_k(\Delta t, 0) + \left(S^\pi_{t_{i-1}} - S^\pi_{t_{i-2}}\right) 2\Delta t \; c^p_k(2\Delta t, 0) + \ldots \\
&\quad + \left(S^\pi_{t_1} - S^\pi_{t_0}\right) i\Delta t \; c^p_k(i\Delta t, 0) + S_0(-1)^k \frac{\sqrt{2p}}{p}.
\end{aligned}$$

□

Remark 3.1.1. We have chosen to present here the case of simple arithmetic averages because it is the most common type of moving average used in practice. However, our numerical method is applicable to any other kind of discrete-time approximation of the moving average process X. In particular, a trapezoidal approximation can be used for a better accuracy of the continuous time average approximation, as proposed by Lapeyre and Temam [76] in the case of Asian options. In such a case, the trapezoidal-type moving average is defined on π by:

$$X^\pi_{t_i} = \frac{1}{N_\delta} \sum_{j=i-N_l-N_\delta}^{i-1-N_l} \frac{S^\pi_{t_j} + S^\pi_{t_{j+1}}}{2}, \quad \forall t_i \in \pi, t_i \geq \delta + l.$$

Instead of (3.5), extending the definition of the discrete-time version S^π of S by

$$S_t^\pi = \frac{S_{t_i}^\pi + S_{t_{i+1}}^\pi}{2}, \forall t \in (t_i, t_{i+1}),$$

the exact discrete-time versions $X^{p,k,\pi}$ of the Laguerre processes are:

$$\begin{aligned} X_{t_i}^{p,k,\pi} &= \sum_{j=2}^{i} \frac{1}{2}\left(S_{t_j}^\pi - S_{t_{j-2}}^\pi\right)(i-j+1)\Delta t \; c_k^p((i-j+1)\Delta t, 0) \\ &\quad + \frac{1}{2}\left(S_{t_1}^\pi - S_0^\pi\right) i\Delta t \; c_k^p(i\Delta t, 0) + S_0(-1)^k \frac{\sqrt{2p}}{p}, \quad \forall t_i \in \pi. \end{aligned}$$

Step 3: Backward resolution of the optimal stopping time problem

The resolution is based on the well-known backward American dynamic programming principle. We adopt a Longstaff and Schwartz-style approach which consists in estimating the optimal exercise time (or equivalently the optimal cashflows generated by the optimal exercise rule) instead of focusing on the computation of the option value processes (as for example in Tsitsiklis and Van Roy [109]). Throughout this Part, the approach presented below, preceded by Step 1 and Step 2 will be called (Lag-LS). The optimal payoffs are evaluated using the *approximate value* of moving average X^π derived from (2.16):

$$M_{t_i}^{n,p_{\text{opt}},\pi} = (1 - H_n^{p_{\text{opt}}}(0)) \; S_{t_i}^\pi + \sum_{k=0}^{n-1} a_k^{p_{\text{opt}}} X_{t_i}^{p_{\text{opt}},k,\pi}, \forall t_i \in \pi \tag{3.7}$$

in which we recall that:

- $H_n^{p_{\text{opt}}}(0) = \sqrt{2p_{\text{opt}}} \sum_{k=0}^{n-1} A_k^{p_{\text{opt}}}$ can be explicitly computed with (2.26) and (2.28),
- so does $a_k^{p_{\text{opt}}}$, $\forall k = 0, \ldots, n-1$ from expression in (2.25).

Besides, one can again use the scaling property from coefficients a_k^p, A_k^p and c_k^p in Proposition 2.3.2 to avoid the computation of these expressions for each given (δ, l).

Denote by $(\tau_i^\pi)_{i=N_\delta,\ldots,N}$ the sequence of discretized optimal exercise times: τ_i^π is the optimal exercise time after $t_i \in \pi$. The backward algorithm works as follows:

1. Initialization: $\tau_N^\pi = T$
2. Backward induction for $i = N-1, \ldots, N_\delta + N_l$:

$$\tau_i^\pi = t_i \mathbb{1}_{A_i} + \tau_{i+1}^\pi \mathbb{1}_{\complement A_i} \text{ with } A_i = \left\{\phi\left(S_{t_i}^\pi, M_{t_i}^{n,p_{\text{opt}},\pi}\right) \geq \mathbb{E}_{t_i}\left[\phi\left(S_{\tau_{i+1}^\pi}^\pi, M_{\tau_{i+1}^\pi}^{n,p_{\text{opt}},\pi}\right)\right]\right\}$$

3. Estimation of the option price at time 0:

$$V_0^\pi = \mathbb{E}\left[\phi\left(S_{\tau_{N_\delta}^\pi}^\pi, M_{\tau_{N_\delta}^\pi}^{n,p_{\text{opt}},\pi}\right)\right]$$

in which:

$$\mathbb{E}_{t_i}[\cdot] = \mathbb{E}\left[\cdot | \left(S^{\pi}_{t_i}, X^{p_{\text{opt}},0,\pi}_{t_i}, \ldots, X^{p_{\text{opt}},n-1,\pi}_{t_i}\right)\right].$$

Estimators of the conditional expectations are constructed with a Monte-Carlo based technique. It consists in using $M \geq 1$ simulated paths on π of the $(n+1)$-dimensional state process:

$$\left(S^{\pi,(m)}, X^{p_{\text{opt}},0,\pi,(m)}, \ldots, X^{p_{\text{opt}},n-1,\pi,(m)}\right), \forall m \leq M.$$

The corresponding paths of the approximate moving average are denoted by

$$M^{n,p_{\text{opt}},\pi,(m)}, \forall m \leq M.$$

Conditional expectations estimators $\mathbb{E}^M_{t_i}$ are then computed by regression on local basis functions (see the precise description of the procedure in Bouchard and Warin [21]). We shall denote by $\left(b^S, b^X_0, \ldots, b^X_{n-1}\right)$ the numbers of basis functions used in each direction of the state variable: b^S for S^{π}, b^X_0 for $X^{p_{\text{opt}},0,\pi}$, b^X_1 for $X^{p_{\text{opt}},1,\pi}$, etc. The Monte-Carlo based backward procedure becomes thus:

1. Initialization: $\tau^{\pi,(m)}_N = T$, $\forall m \leq M$
2. Backward induction for $i = N-1, \ldots, N_\delta + N_l$, $\forall m \leq M$:

$$\begin{cases} \tau^{\pi,(m)}_i = t_i \mathbb{1}_{A^{(m)}_i} + \tau^{\pi,(m)}_{i+1} \mathbb{1}_{\complement A^{(m)}_i} \\ A^{(m)}_i = \left\{\phi\left(S^{\pi,(m)}_{t_i}, M^{n,p_{\text{opt}},\pi,(m)}_{t_i}\right) \geq \mathbb{E}^M_{t_i}\left[\phi\left(S^{\pi}_{\tau^{\pi}_{i+1}}, M^{n,p_{\text{opt}},\pi}_{\tau^{\pi}_{i+1}}\right)\right]\right\} \end{cases}$$

3. Estimation of the option price at time 0:

$$V^{\pi}_0 = \frac{1}{M}\sum_{m=1}^{M} \phi\left(S^{\pi,(m)}_{\tau^{\pi,(m)}_{N_\delta}}, M^{n,p_{\text{opt}},\pi,(m)}_{\tau^{\pi,(m)}_{N_\delta}}\right)$$

Remark 3.1.2. We will use a numerical improvement to this standard backward induction algorithm, which might seem rather natural for practitioners. It consists in evaluating the optimal cashflows by using the *exact value* (3.6) of the moving average. In particular, the optimal stopping frontier becomes:

$$A^*_i = \left\{\phi\left(S^{\pi}_{t_i}, X^{\pi}_{t_i}\right) \geq \mathbb{E}_{t_i}\left[\phi\left(S^{\pi}_{\tau^{\pi}_{i+1}}, X^{\pi}_{\tau^{\pi}_{i+1}}\right)\right]\right\}.$$

This improved method will be called (Lag-LS*) and we will show a monotone convergence in n on the contrary to (Lag-LS), see Section 4.1.3 of next Chapter 4.

3.2 Reference methods

Benchmark method With the same notations as in Section 3.1, recall that the dimension of the discrete-time version of moving average option pricing problem (3.1) is equal to $(N_\delta + N_l)$ with a Markovian state:

$$\left(S^{\pi}_{t_i}, S^{\pi}_{t_{i-1}}, \ldots, S^{\pi}_{t_{i-N_l-N_\delta+1}}\right), \forall t_i \in \pi, t_i \geq \delta + l.$$

The larger the length of the averaging time window δ and the time lag l, the higher the dimension of the state vector. We use the standard Longstaff and Schwartz algorithm for such a Bermudan option in dimension $(N_\delta + N_l)$ as the benchmark method. This method will be called (M-LS) and our Monte Carlo regression based approach (see more details in [21]) allows to deal with cases *up to dimension* 8. For applications in which $(N_\delta + N_l)$ is larger, this method becomes computationally unfeasible.

Lower reference: a "non Markovian" approximation for moving average options Motivated by a reduction of dimensionality, the numerical method that is most often used in practice to value moving average options consists in computing the conditional expectations in the Longstaff-Schwartz algorithm using only the explanatory variables (S, X): namely, the price and the moving average appearing in the option payoff. The resulting exercise time is thus *suboptimal*, but the approximate option price is often close to the true price. To assess the improvement offered by our method, in Chapter 4, we will systematically compare our approximation to this suboptimal approximate price, also computed using a Longstaff and Schwartz approach and referred to as (NM-LS).

Let $(\theta_i^\pi)_{i=N_\delta,\dots,N}$ denote the discrete-time sequence of the estimated optimal exercise times (θ_i^π being the optimal exercise time after $t_i \in \pi$), (NM-LS) works as follows:

1. Initialization: $\theta_N^\pi = T$
2. Backward induction for $i = N-1, \dots, N_\delta + N_l$:
$$\theta_i^\pi = t_i \mathbb{1}_{A_i} + \theta_{i+1}^\pi \mathbb{1}_{\complement A_i} \text{ with } A_i = \left\{ \phi\left(S_{t_i}^\pi, X_{t_i}^\pi\right) \geq \mathbb{E}\left[\phi\left(S_{\theta_{i+1}^\pi}^\pi, X_{\theta_{i+1}^\pi}^\pi\right) \mid \left(S_{t_i}^\pi, X_{t_i}^\pi\right) \right] \right\}$$
3. Estimation of the option price at time 0:
$$U_0^\pi = \mathbb{E}\left[\phi\left(S_{\theta_{N_\delta}^\pi}^\pi, X_{\theta_{N_\delta}^\pi}^\pi\right) \right]$$

Similarly to other methods, the conditional expectations are computed with the adaptative local basis regression-based technique from [21]. The numbers of basis functions used in each direction will be denoted by b^S for S^π and b^X for X^π.

3.3 Some extensions

3.3.1 Dealing with multi-assets based moving averages

Let us consider the multi-assets framework introduced in Remark 2.1.1. The pricing problem is the following:

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau^1, S_\tau^2, \dots, S_\tau^d, X_\tau\right) \right], \tag{3.8}$$

in which X is indexed on moving averages of $\bar{d}$ asset prices, say $\{S^1, S^2, \dots, S^{\bar{d}}\}$:

$$X_t = K + \sum_{j=1}^{\bar{d}} \alpha_j \left(\frac{1}{\delta} \int_{t-l-\delta}^{t-l} S_u^j du \right), \quad \forall t \geq \delta + l$$

and $K \geq 0$ is a fixed strike price.

► The extension of solving methods (M-LS) and (NM-LS) to this framework is straightforward. On one hand, the dimension of the discrete-time version of problem (3.8) is equal to

$$\bar{d} \times (N_\delta + N_l) + (d - \bar{d})^+,$$

so that (M-LS) is actually computationally unfeasible in practice. On the other hand, the non Markovian approximate method (NM-LS) uses as explanatory variables (S^1, S^2, S^d, X) for computing the conditional expectation estimators.

► The application of the same Laguerre approximation as the one introduced in Chapter 2 is straightforward if you write X as

$$X_t = \frac{1}{\delta}\int_{t-l-\delta}^{t-l}\left(K + \sum_{j=1}^{\bar{d}} \alpha_j S_u^j\right) du, \quad \forall t \geq \delta + l.$$

The resulting finite-dimensional approximation to problem (3.8) is

$$\sup_{\tau \in \mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[\phi\left(S_\tau^1, S_\tau^2, \ldots, S_\tau^d, M_\tau^{n,p}\right)\right] \tag{3.9}$$

in which:

- the moving average X is approximated by

$$M_t^{n,p} = (1 - H_n^{p_{\text{opt}}}(0))\,\big(K + \sum_{j=1}^{\bar{d}} \alpha_j S_t^j\big) + \sum_{k=0}^{n-1} a_k^p X_t^{p,k}, \quad \forall t \geq 0,$$

- the Laguerre processes are defined by:

$$X_t^{p,k} = \int_0^\infty L_k^p(v)\big(K + \sum_{j=1}^{\bar{d}} \alpha_j S_{t-v}^j\big) dv, \quad \forall t \geq 0, \forall k = 0, \ldots, n-1.$$

As noticed in Remark 2.2.1, the result from Corollary 2.2.1 still holds. Problem (3.9) constitutes a $(d \vee \bar{d} + n)$-dimensional approximation to infinite dimensional problem (3.8) with a Markovian state

$$(S^1, S^2, \ldots, S^{d\vee\bar{d}}, X^{p,0}, X^{p,1}, \ldots, X^{p,n-1}).$$

Indeed, the same arguments as the one used in the proof for Proposition 2.2.1 leads to the following dynamics of the Laguerre processes $(X^{p,k})_{k\geq 0}$:

$$\begin{cases} dX_t^{p,0} = \left(\sqrt{2p}\left(K + \sum_{j=1}^{\bar{d}} \alpha_j S_t^j\right) - pX_t^{p,0}\right) dt \\ dX_t^{p,1} = \left(\sqrt{2p}\left(K + \sum_{j=1}^{\bar{d}} \alpha_j S_t^j\right) - 2pX_t^{p,0} - pX_t^{p,1}\right) dt \\ \vdots \\ dX_t^{p,n-1} = \left(\sqrt{2p}\left(K + \sum_{j=1}^{\bar{d}} \alpha_j S_t^j\right) - 2p\sum_{k=0}^{n-2} X_t^{p,k} - pX_t^{p,n-1}\right) dt. \end{cases}$$

The extension of the solving methods (Lag-LS) and (Lag-LS*) is then immediate (in particular Step 1 remains unchanged).

3.3.2 Dealing with moving averages updated at fixed times

Indexed gas supplying contracts have often the following specific characteristic: the strike price X is not updated continuously but only at some fixed updating times (which will be called *fixing dates*). Assume for the sake of simplicity that these fixing dates are uniformly distributed on $[0, T]$. Let q be the length of the strike validity period. The strike price implied in the option payoff is no more X but

$$\bar{X}_t = X_{\varphi_q(t)} \text{ with } \varphi_q(t) := q\left\lfloor \frac{t}{q} \right\rfloor, \quad \forall t \geq \delta + l,$$

in which $\varphi_q(t)$ corresponds to the last fixing date before t.

Let us consider that the fixing time grid is a sub-set of the discrete-time resolution grid π (it is the case in practice: possible exercise times of the Bermudan option are necessarily strike updating times). Then, the extension of the solving algorithms (M-LS), (NM-LS), (Lag-LS) and (Lag-LS*) is straightforward by using the following idea: to keep the system Markovian in this framework, you just need to add $\bar{X}$ to the state vector. We briefly sum up in the paragraphs below what to modify in each algorithm.

► (M-LS): add a dimension to stay in a Markovian framework.
The Markovian state is $(N_\delta + N_l + 1)$-dimensional:

$$\left(S^\pi_{t_i}, S^\pi_{t_{i-1}}, \ldots, S^\pi_{t_{i-N_l-N_\delta+1}}, \bar{X}^\pi_{t_i}\right). \tag{3.10}$$

Indeed, at each time t_i belonging to a strike validity period i.e. such that $\varphi_q(t_i) < t_i$, $\bar{X}^\pi_{t_i} = \bar{X}^\pi_{t_{i-1}} = X^\pi_{\varphi_q(t_i)}$. At each date t_i corresponding to a fixing time ($\varphi_q(t_i) = t_i$), one can compute the strike price as

$$\bar{X}^\pi_{t_i} = \frac{1}{N_\delta} \sum_{j=i-N_l-N_\delta+1}^{i-N_l} S^\pi_{t_j}.$$

Because of the collinearity between $\bar{X}^\pi_{t_i}$ and the N_δ components $\left(S^\pi_{t_{i-N_l}}, \ldots, S^\pi_{t_{i-N_l-N_\delta+1}}\right)$ of the Markovian state in this last case, only the first $(N_\delta + N_l)$ components of (3.10) are used as regression variables for computing the conditional expectation estimators.

► (NM-LS): use $(S, \bar{X})$ as explanatory variables for computing the conditional expectation estimators.

► (Lag-LS): in a same way, this solving algorithm is modified by adding a dimension. The Laguerre approximation of the strike is updated at the fixing dates:

$$\bar{M}^{n,p_{\text{opt}}}_t = \bar{M}^{n,p_{\text{opt}}}_{\varphi_q(t)}.$$

This leads to a $(n+2)$-dimensional approximation with Markovian state:

$$\left(S_{t_i}, X^{p_{\text{opt}},0}_{t_i}, X^{p_{\text{opt}},1}_{t_i}, \ldots, X^{p_{\text{opt}},n-1}_{t_i}, \bar{M}^{n,p_{\text{opt}},\pi}_{t_i}\right).$$

Indeed, at each time t_i belonging to a strike validity period, $\bar{M}^{n,p_{\text{opt}},\pi}_{t_i} = \bar{M}^{n,p_{\text{opt}},\pi}_{t_{i-1}} = \bar{M}^{n,p_{\text{opt}},\pi}_{\varphi_q(t_i)}$. At each date t_i corresponding to a fixing time, i.e. such that $\varphi_q(t_i) = t_i$, one can compute the strike approximate value as:

$$\bar{M}^{n,p_{\text{opt}},\pi}_{t_i} = \left(1 - H^{p_{\text{opt}}}_n(0)\right) S^\pi_{t_i} + \sum_{k=0}^{n-1} a^{p_{\text{opt}}}_k X^{p_{\text{opt}},k,\pi}_{t_i}.$$

Again, because of the collinearity between $\bar{M}^{n,p_{\text{opt}},\pi}_{t_i}$ and the first n components of the Markovian state in this last case, only these n components are used as regression variables for computing the conditional expectation estimators.

► (Lag-LS*): improve (Lag-LS) by using $\bar{X}$ instead of $\bar{M}^{n,p_{\text{opt}}}$.

Chapter 4

Numerical applications

We present numerical results obtained with the methods described in previous Chapter 3 for pricing moving average Bermudan-style options. We include a numerical study of the previously-introduced approximation by Laguerre expansion of moving average processes as well. The tests are performed in two frameworks: firstly for moving average options pricing in a one-asset Black and Scholes framework, secondly for the valuation of oil-indexed supplying contracts commonly encountered in the gas market.
For all the notations which are used in present chapter, we refer the reader to Chapter 3, in which they already have been introduced.

In Section 4.1, numerical experiments are performed in a one-dimensional Black and Scholes framework. We find first that for standard moving average American options (without time delay), the error made by the non Markovian approximate method (NM-LS) is not so large (less than 1% for the examples we took). This justifies the use of this approach for practical purposes in spite of its suboptimality. To our best knowledge, no result is available concerning the error made by such an approximation (see e.g., Bilger [15] and Grau [57]).

We then perform simulation tests which demonstrate the efficiency and accuracy of the Laguerre-based approach for approximating moving average processes, delayed in time or not. Around $n = 3$ functions in the Laguerre series (for a zero time delay $l = 0$) and $n = 5$ functions (when $l > 0$) are sufficient to provide very accurate dynamics approximation. Besides, this holds whatever the values of the averaging period and the time delay l.

In a second time, a convergence study on n leads us to prefer (Lag-LS*) to (Lag-LS) for pricing moving average options: this first algorithm shows a monotone convergence on n and gives stable and converged option prices already with around $n = 3$ Laguerre functions. This allow us to perform a more general study of moving average options price as function of the averaging window and the time delay, which, to our best knowledge, has never been done before. For moving average American options with large time delay, (Lag-LS*) gives option's prices up to 11% above the suboptimal prices given by (NM-LS).

On the other hand, we observe that our method (Lag-LS*) converges faster than the classical method (M-LS) with respect to the state dimension for pricing a same moving average Bermudan option. Finally, we compare the pricing results obtained by (Lag-LS*) to some relevant moving average option prices that we have found in the literature [22, 57].

In Section 4.2, our method (Lag-LS*) is used for pricing Bermudan-style contracts in the European gas market, including relatively realistic characteristics. The strike prices of such

contracts are commonly functions of moving averages of various commodity prices: we deal with gas oil and fuel oil-indexed strike prices.

As shown on some simulations, the Laguerre expansion-based approach remains very accurate for approximating those stochastic strikes. And yet, on the practical examples we took, our method gives contract values very close (less than 0.5% above) to the non Markovian approximate method (NM-LS) used in practice. This is mainly due to monthly-updated strike prices, large averaging windows (up to 6 months for one year contracts) and relatively small time delays, and probably the mean-reverting behavior of gas and oil prices.

4.1 Experiments in the Black and Scholes framework

Consider a moving average American option with time delay with value at time 0:

$$\sup_{\tau\in\mathcal{T}_{[\delta+l,T]}} \mathbb{E}\left[e^{-r\tau}\phi\left(S_\tau, X_\tau\right)\right], \quad X_\tau = \frac{1}{\delta}\int_{t-l-\delta}^{t-l} S_u du,$$

where the asset price S is assumed to follow the risk-neutral Black and Scholes dynamics:

$$dS_t = S_t\left(rdt + \sigma dW_t\right), \quad S_0 = s, \tag{4.1}$$

in which W is a standard Brownian motion. and W is a standard Brownian motion. We shall consider options with payoff $\phi(s,x) = (s-x)^+$ (call option) or $\phi(s,x) = (x-s)^+$ (put option). Unless specified otherwise, the following parameters are used throughout this Section:

Maturity	$T = 0.2$
Risk free interest	$r = 5\%$
Volatility	$\sigma = 30\%$
Initial spot value	$s = 100$

and we consider a Bermudan option with exercise possible every day (when $T = 0.2$, the time interval $[0,T]$ is divided into $N = 50$ time steps).

4.1.1 Reference methods for pricing moving average options

Set $l = 0$. Table 4.1 shows the prices of moving average call options computed by (NM-LS) and (M-LS)[1] for various averaging periods δ (recall $N_\delta = \delta/\Delta t$), with $M = 10$ million of Monte Carlo paths and $b^S = b^X = 2$. The prices are averages over 5 valuations and the relative standard deviation is given in brackets.

For reasonable volatility coefficients of the underlying price process and relatively small averaging window δ, (NM-LS) seems to provide a very good approximation (from below) to moving average options prices. This justifies the approximation made by practitioners and (among others) by Broadie and Cao [22]. This might be explained by the fact that, for small averaging windows, the information loss implied by (NM-LS) (which does take into account the path dependence of the cashflows, namely the asset prices over the averaging period) is negligible. Our Laguerre approximation-based pricing method will allow us to verify if this still holds when increasing δ and adding a time delay.

[1] Our least squares Monte Carlo based-algorithms allow us to deal with state vectors up to dimension 8.

N_δ	(NM-LS)	(M-LS)	Relative error
2	1.890 (0.011 %)	1.890 (0.011 %)	0.00 %
3	2.684 (0.011 %)	2.685 (0.010 %)	0.05 %
4	3.183 (0.018 %)	3.186 (0.012 %)	0.10 %
5	3.526 (0.016 %)	3.531 (0.007 %)	0.15 %
6	3.773 (0.016 %)	3.780 (0.013 %)	0.19 %
7	3.955 (0.011 %)	3.964 (0.215 %)	0.22 %
8	4.092 (0.015 %)	4.103 (0.316 %)	0.28 %
9	4.193 (0.016 %)		
10	4.268 (0.019 %)		

Table 4.1: Moving average options pricing with (NM-LS) and (M-LS).

Remark 4.1.1. Since the variance of the moving average process X is much smaller than the variance of the spot price S, increasing the number b^X of basis functions in the direction of state X has a negligible impact on the conditional expectation estimations and the resulting pricing results given by (NM-LS). On the contrary, the number b^S of basis functions in the direction of S should be sufficiently large.

We report in Figure 4.1 the prices of moving average options (call and put) computed by (NM-LS) when varying the averaging period δ from 0 to T (we used $M = 5$ million, $b_S = 3$ and $b_X = 1$). In the limit case when $\delta = T$, we retrieve the price of the Asian option with payoff $\phi(S_T, \frac{1}{T}\int_0^T S_t dt)$: exact value equal to 3.29 for the call option and to 2.79 for the put option.

In particular, the *change of monotony* of the option prices as functions of δ may be explained as follows: the option price is maximized for an averaging period sufficiently large to increase the distance between the spot price and its moving average, but at the same time the early exercise feature implies an arbitrage with respect to time to maturity, hence the difference for call and put options.

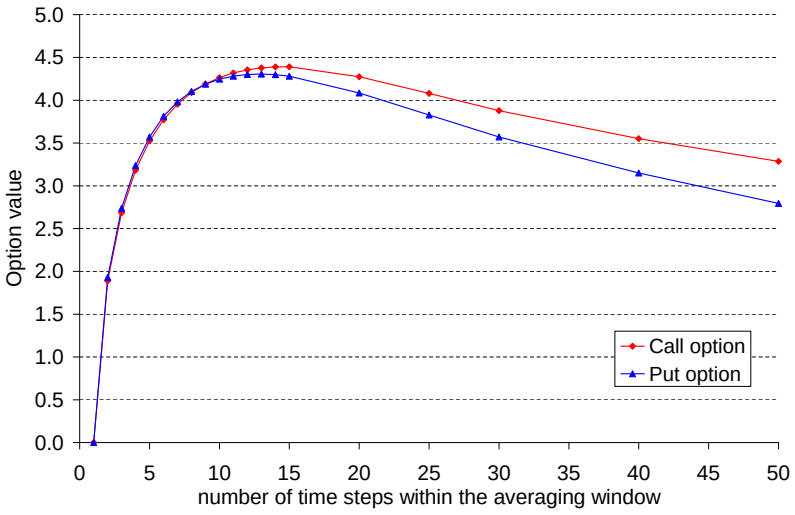

Figure 4.1: Moving option prices as function of the averaging window.

For pricing options with large averaging window and/or time lag, due to a high dimension, (M-LS) becomes computationally unfeasible[1]. Therefore, (NM-LS) is here the only benchmark (lower bound) to which we can refer. In the sequel, we shall systematically compare the option prices computed with our Laguerre approximation-based method to the suboptimal price computed by (NM-LS), to assess the improvement offered by our method.

4.1.2 Laguerre approximation of moving average processes

Standard moving averages

Figure 4.2 shows a simulated trajectory of the underlying price S^π, its moving average X^π with $\delta = 0.04$ ($N_\delta = 10$) and the corresponding Laguerre-based moving average approximation $M^{n,p_{\text{opt}},\pi}$ with $n = 1, 3$ and 7 Laguerre basis functions. Already for $n \geq 3$ Laguerre basis functions, $M^{n,p_{\text{opt}},\pi}$ accurately mimics the exact moving average dynamics of X^π and this approximation seems to be almost exact when $n = 7$.

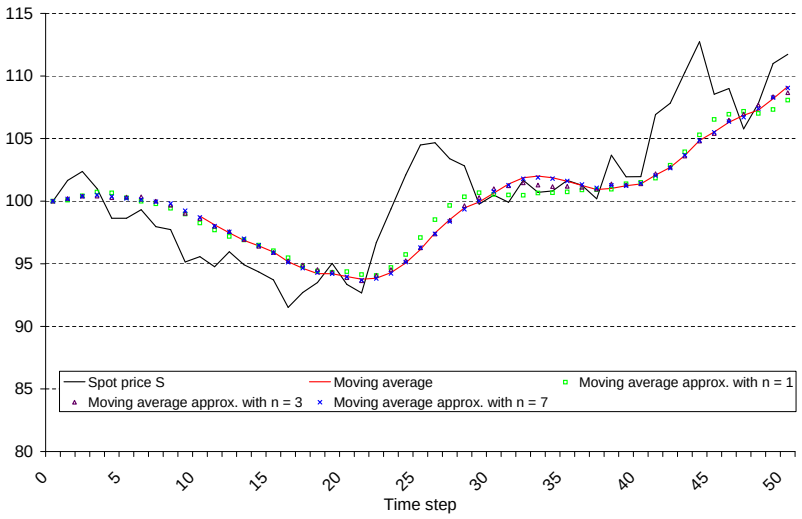

Figure 4.2: Simulated trajectory ot the moving average process and its Laguerre approximation.

Figures 4.3 and 4.4 illustrate the fact that this good accuracy holds true when varying δ and inscreasing the volatility σ of the underlying price process. In particular, Figure 4.4 highlights how our the Laguerre-based approximate processes $M^{n,p_{\text{opt}},\pi}$ oscillate around the exact moving average process X^π.

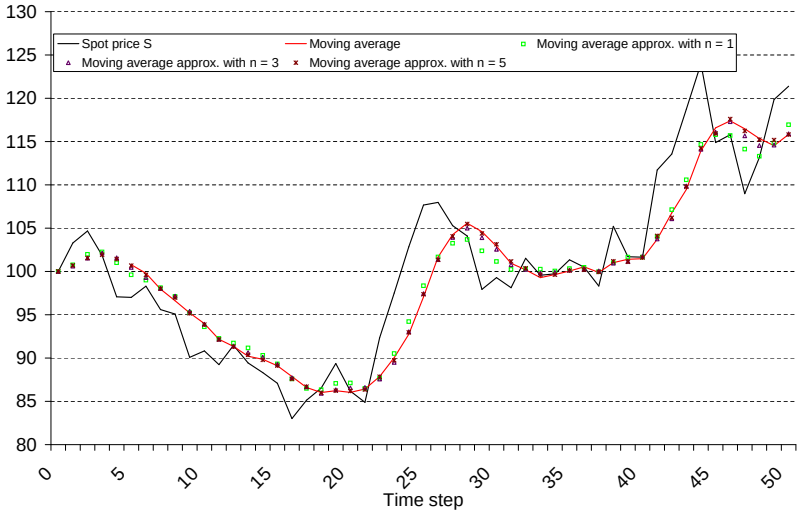

Figure 4.3: Trajectories when $(\delta, \sigma) = (0.02, 60\%)$.

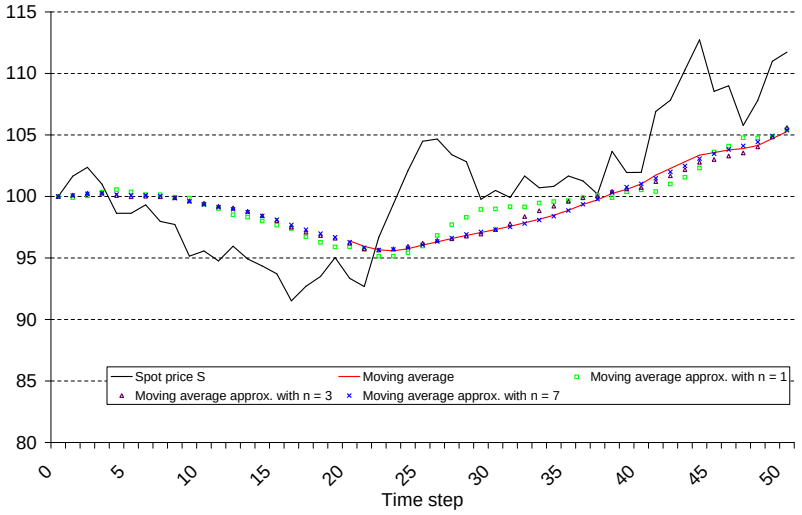

Figure 4.4: Trajectories when $(\delta, \sigma) = (0.08, 30\%)$.

Moving averages with time delay

Consider now moving averages with a time delay ($l > 0$). We report in Figures 4.5 and 4.6 some Laguerre-based approximations of the delayed moving average X^π when $(\delta, l) = (0.04, 0.02)$ and $(\delta, l) = (0.02, 0.04)$ respectively.

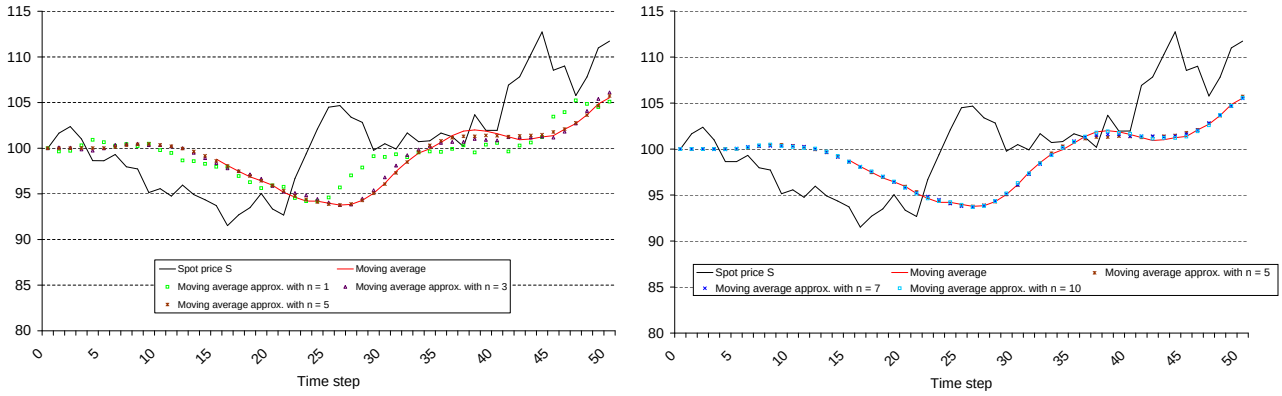

Figure 4.5: Laguerre approximation of the moving average with time delay ($\delta = 2l$).

As expected (see the Step 1 in Section 3.1 of previous Chapter), more Laguerre functions are required to accurately mimic a delayed moving average process: compare Figure 4.5 to its counterpart with a zero time lag in Figure 4.2. This effect is intensified when $\delta < l$ as shown in Figure 4.6.

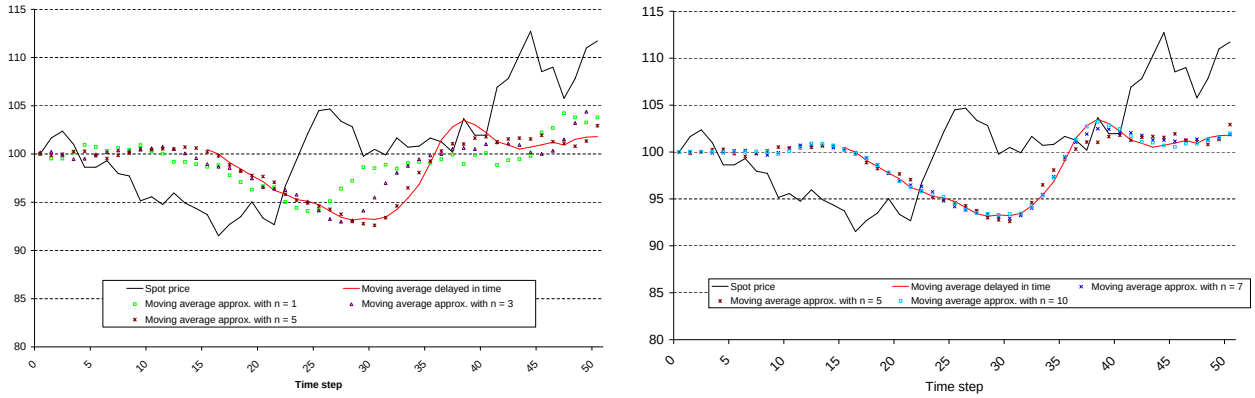

Figure 4.6: Laguerre approximation of the moving average with time delay ($l = 2\delta$).

These numerical observations enforce the main interest of our approach: our Laguerre approximation of moving average processes - using an optimal scaling of the Laguerre functions - is scale invariant: the error made is stable when changing the length of the averaging window δ. For simple moving averages (without any time lag) around $n = 3$ Laguerre functions are sufficient to obtain accurate approximations, *whatever the value of* δ. Our Laguerre approximation captures less easily delayed moving averages dynamics ($n \sim 5$ Laguerre functions required) and even less when $\delta < l$ ($n \sim 7$).

4.1.3 Moving average options pricing with the Laguerre approximation

Standard moving average options

Consider a zero time-lag and an averaging period $\delta = 0.04$. Table 4.2 reports the prices of moving average call options computed[2] using the Laguerre-based method (Lag-LS) and its improved

[2]Our numerical methods (Lag-LS) and (Lag-LS*) can handle with state vectors up to dimension 8. In present one-dimensional asset framework, the dimension of the state vector is equal to $(n + 1)$. This allows us to use at the most $n = 7$ Laguerre functions.

version (Lag-LS*). The price values are means over 5 valuations, the relative standard deviation is given in brackets and we used $M = 5$ million Monte Carlo paths for $n = 1, \dots, 3$ Laguerre functions and $M = 10$ million Monte Carlo paths for $n = 4, \dots, 7$ Laguerre functions, with $b^S = 4$ and $b_k^X = 1, \forall k \geq 0$. With $M = 10$ million Monte Carlo paths, $b^S = 4$ and $b^X = 1$, (NM-LS) gives an option value equal to 4.268.

n	(Lag-LS*)	(Lag-LS)
1	4.266 (0.020 %)	4.092 (0.017 %)
2	4.273 (0.022 %)	4.302 (0.019 %)
3	4.276 (0.023 %)	4.182 (0.018 %)
4	4.276 (0.022 %)	4.227 (0.020 %)
5	4.277 (0.023 %)	4.275 (0.020 %)
6	4.277 (0.024 %)	4.287 (0.022 %)
7	4.277 (0.024 %)	4.258 (0.022 %)

Table 4.2: Moving average options pricing with (Lag-LS) and (Lag-LS*).

Whereas (Lag-LS) oscillates as n increases (this is due to the *non monotone approximation* of the moving average X by $M^{n,p_{\text{opt}}}$, see for example Figure 4.4), (Lag-LS*) shows a monotone convergence when increasing n, as shown in Figure 4.7. This behavior holds in all the numerical experiments that we have performed (including various option's payoff). Intuitively, this comes from the fact that when increasing n, some information is added and the resulting conditional expectations estimations tend to their exact value from below. In this example, the limiting value given by (Lag-LS*) (almost 4.277) is around 0.2% above the benchmark value given by (NM-LS).

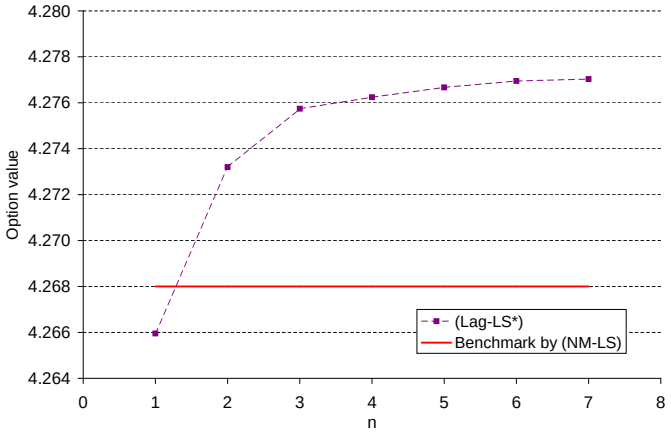

Figure 4.7: Convergence of the improved Laguerre-based approximation.

Remark 4.1.2. When the averaging window is large, the variance of the Laguerre states $(X_k^{p_{\text{opt}},k})_{k\geq 0}$ is small, and at least much smaller than the variance of the price S, as shown in Figure 4.8 with $n = 5$. In consequence, increasing the numbers b_k^X of basis functions in the directions of these states does not have a strong impact on the conditional expectation estimators and the resulting option price. On the contrary, the number b^S of basis functions in the direction of the spot price S should be sufficiently large.

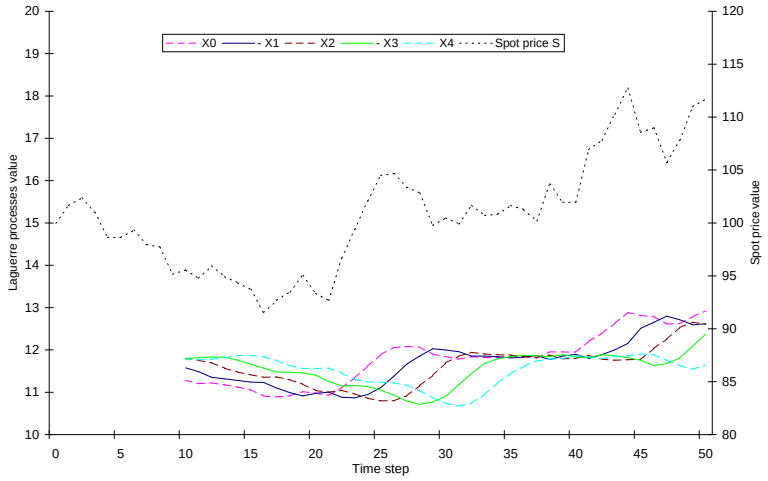

Figure 4.8: Simulated trajectory of the asset price and corresponding Laguerre processes.

Figure 4.9 presents the prices of moving average call options computed by (Lag-LS*) and (NM-LS) when varying δ from 0 to T with the same parameters as above. 7 Laguerre basis functions were used with method (Lag-LS) as soon as $N_\delta \geq 8$. For smaller N_δ, we take $n = N_\delta - 1$: n must satisfy the condition $n \leq N_\delta - 1$ because otherwise the estimation of the conditional expectation at time t_{N_δ} leads to a degenerate linear system. For large averaging periods, the price that we obtain with 7 Laguerre functions is about 0.3% above the lower bound given by (NM-LS).

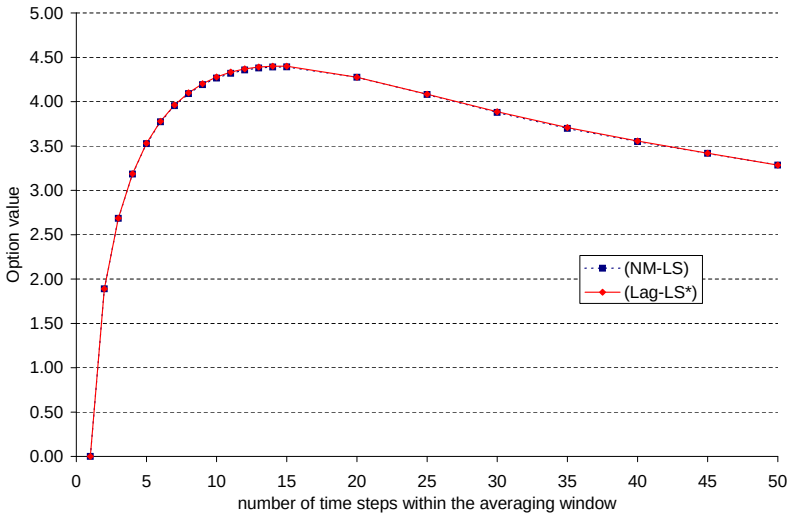

Figure 4.9: Moving average option price as function of the averaging window δ.

Already with $n = 2$ Laguerre functions, our method (Lag-LS*) gives *greater prices* than the (suboptimal) non Markovian method (NM-LS). In all the numerical experiments that we performed (changing the payoff of the option and varying the model parameters), we retrieved

this same behavior. And yet, even for large averaging periods, the relative error between (Lag-LS*) (with $n = 7$) to the lower reference (NM-LS) is not so large (less than 1% for all the examples we took). In this sense, the 2-dimensional method (NM-LS) constitutes a very good approximation for pricing standard moving average options: it requires less computational effort than (Lag-LS*) for an equivalent accuracy.

Moving average options with time delay

With the same option characteristics and parameters as above and an averaging period equal to $\delta = 0.02$ (number of time steps $N_\delta = 5$), Figure 4.10-(left) presents the prices of delayed moving average call options computed by (Lag-LS*) and (NM-LS) when varying l from 0 to $T - \delta$. In the limit case when $l = T - \delta$, we retrieve the price of the Asian option with payoff $\left(S_T - \frac{1}{T-l}\int_0^{T-l} S_t dt\right)^+$ (exact value equal to 5.60).

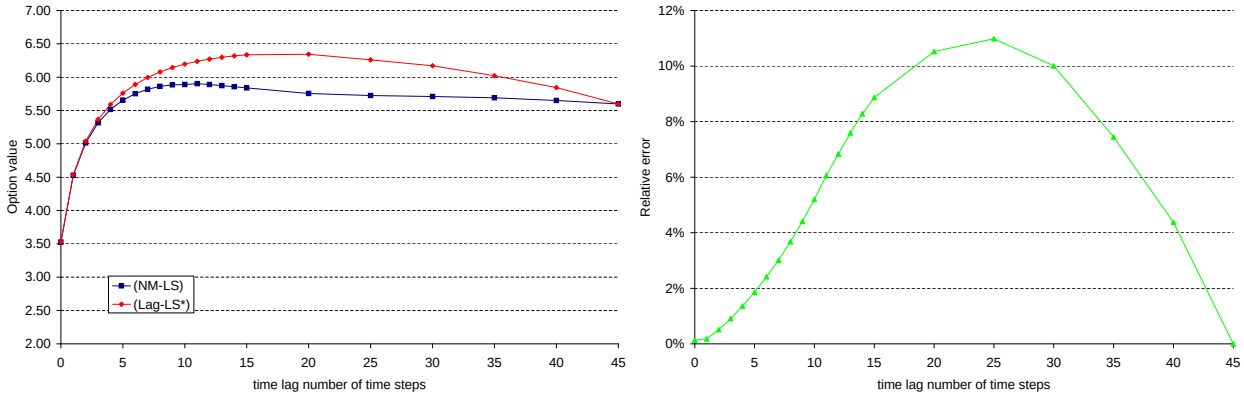

Figure 4.10: Prices of moving average call options with time delay as function of the lag l (left) and relative pricing error between (NM-LS) and (Lag-LS*) (right).

The relative difference between the option values given by (Lag-LS*) and (NM-LS) is significant (bigger than 5%) for time lags such that $l \in [0.04, 0.152]$ (corresponding to $10 \leq N_l \leq 38$), see Figure 4.10-(right). For example, when $l = 0.1$ (corresponding to $N_l = 25$ time steps), the relative difference is around 11%.

Now fix $l = 0.08$ ($N_l = 20$). As shown in Figure 4.11, when the averaging window increases this relative difference decreases. But it is still around 5% when $N_\delta = 15$. Again, already with $n = 2$ Laguerre basis functions, our method gives greater prices than (NM-LS). We retrieve these behaviors in other numerical tests performed with different option's payoffs.

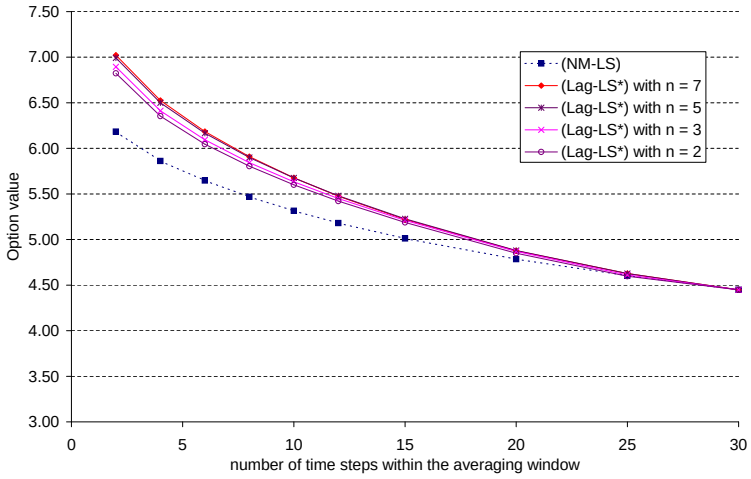

Figure 4.11: Price of the moving average option with time delay as function of the averaging window δ.

For moving average American options with time delay, whose payoff depends on the average of the price between dates $t-l$ and $t-l-\delta$, the suboptimal approximation (NM-LS) leads to a *significant bias*, in particular when the averaging period δ is small and the time lag l is large.

Pricing of moving average Bermudan options: a convergence rate improvement

Finally, we numerically observe that our method (Lag-LS*) converges faster than classical method (M-LS) with respect to the state dimension for pricing a same Bermudan option.

Let us consider a moving average call option with maturity $T = 0.5$ and moving window $\delta = 0.1$. Figure 4.12 provides a comparison between pricing values given by (Lag-LS*) for a time step $\Delta t = \frac{1}{80}$ when varying the number of Laguerre functions from 1 to 7 and by (M-LS) when varying the number of time steps within the averaging period from 2 to 8, that is $\Delta t = \frac{1}{20}, \frac{1}{30}, \ldots, \frac{1}{80}$ (the state dimension varies in both cases from 2 to 8). We used $M = 20$ million of Monte Carlo paths and $b^S = 2$ for (M-LS), and $M = 15$ million, $b^S = 2$ and $b_k^X = 1$ for (Lag-LS*).

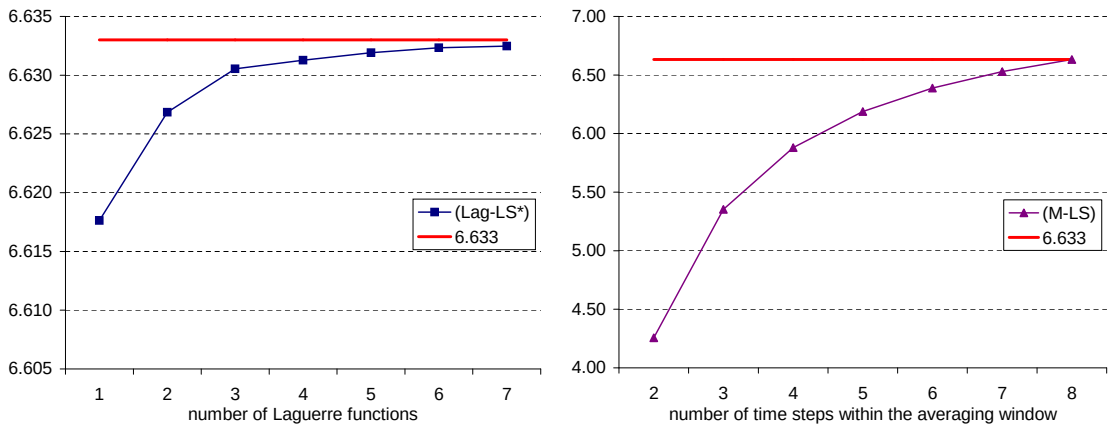

Figure 4.12: Compared convergence of the improved Laguerre-based approximation and the benchmark method for pricing a Bermudan option.

4.1.4 Comparision to some results available in the literature

To our best knowledge, in the literature, only few authors provide numerical results for moving average American options [57, 22].

A moving window Asian option with floating strike

Grau [57] deal with a moving window Asian call option with floating strike in the risk-neutral Black and Scholes framework, see (4.1). The parameters are the following:

Maturity	$T = 0.4$
Payoff	$\phi(s,x) = (x-s)^+$
Length of the moving window	$\delta = 0.04$
Risk free interest	$r = 5\%$
Volatility	$\sigma = 40\%$
Initial spot value	$s = 100$

The converged price given in [57] of the corresponding Bermudan option with

Number of time steps within $[0,T]$	$N = 100$
Number of time steps within the moving window	$N_\delta = 10$

is around 7.60. Table 4.3 presents the prices computed by (Lag-LS*) with $M = 15$ million of Monte Carlo paths, $b^S = 3$ and $b_k^X = 1$.

n	(Lag-LS*)	Relative difference to 7.60
1	7.516	-1.10%
2	7.528	-0.95%
3	7.532	-0.90%
4	7.533	-0.88%
5	7.534	-0.87%
6	7.534	-0.87%

Table 4.3: Pricing a moving average option, comparison to Grau [57].

The numerical method used by Grau is based on the Tsitsiklis and Van Roy [109] approach, which is known to give an estimation from above to Bermudan options prices (see e.g., Bouchard and Warin [21]), on the contrary to the Longstaff and Schwartz-approach implied in our method (Lag-LS*).

Whereas Grau uses a classical Markovian resolution in dimension $N_\delta = 10$, our method requires less computational effort, since it provides a converged and accurate estimation in dimension $(n+1) = 4$.

A moving window Asian option with fixed strike

Consider finally a moving window Asian call option with fixed strike in the Black and Scholes model as Broadie and Cao [22]. A non Markovian approximation is used in [22] for the computation of the conditional expectations in a Longstaff and Schwartz-type algorithm (cf. the choice of regression basis functions in Appendix A.2). The spot price S is written as in (4.1), the option characteristics are

Maturity	$T = 1$
Strike	$K = 100$
Payoff	$\phi(s,x) = (x-K)^+$
Length of the moving window	$\delta = 0.2$
Risk free interest	$r = 5\%$
Volatility	$\sigma = 20\%$

and the discrete-time setting used in [22] is such that:

Number of time steps within $[0,T]$	$N = 50$
Number of time steps within the moving window	$N_\delta = 10$

We report in Figure 4.13 the results obtained with (Lag-LS*) (with $M = 20$ million, $b^S = 4$ and $b_k^X = 1$) when initial spot price $s = 100$. The limiting value (almost 11.39) is 0.11% above the lower bound 11.378 given by Broadie and Cao.

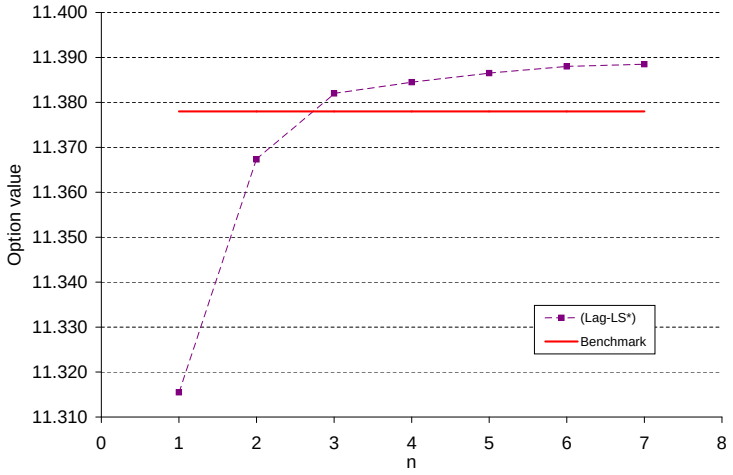

Figure 4.13: Pricing a moving window Asian option with fixed strike, comparison to Broadie and Cao [22].

Table 4.4 shows the option prices computed by (Lag-LS*) with $n = 7$ Laguerre functions and the same parameters as abovewhen varying initial spot price s. Since we do not improve (Lag-LS*) by any variance reduction technique, the best accuracy is obtained for in the money call options.

s	Value given in [22]	(Lag-LS*)	Relative difference
100	11.378	11.389	0.10%
110	19.918	19.950	0.16%
120	29.899	30.004	0.35%
130	40.389	40.732	0.85%

Table 4.4: Pricing a moving average option, comparison to Broadie and Cao [22].

4.2 Valuation of oil-indexed gas contracts

We consider the valuation problem of a gas supplying contract, which allows to its holder to purchase an unitary amount of gas at some oil-indexed strike price, only once before an expiration date. In comparison to Swing options presented in Chapter 2 of Part I (including a multiple exercise feature), the following numerical experiments constitute a preliminary study. For numerical reasons (slow computation speed), we did not handle with this last case, see Remark 4.2.2.

4.2.1 Contracts characteristics

We shall consider American-style contracts whose value at time 0 is given by the solution to:

$$\sup_{\tau \leq T} \mathbb{E}\left[\left(S^g_\tau - \bar{X}_\tau\right)^+\right],$$

in which:

- the time maturity $T = 1$ year (from October to October).
- S^g denotes the gas spot price.
- the oil-indexed strike price has the form:

$$\bar{X}_t = \alpha_{go}\bar{X}_t^{go} + \alpha_{fo}\bar{X}_t^{fo},$$

where $\bar{X}^{go}$ and $\bar{X}^{fo}$ are respectively some moving averages of the gas oil and fuel oil spot prices denoted by S^{go} and S^{fo}. The weights assigned to each component are taken such that:

$$\alpha_{go} = 0.025, \quad \alpha_{fo} = 0.030.$$

- the gas oil and fuel oil prices are averaged on a period with length δ delayed in time with a lag l and the resulting index $\bar{X}^{go}$ and $\bar{X}^{fo}$ are updated only every q months. We refer the reader to Chapter 2 in Part I for more details on these three caracteristic numbers (δ, l, q) of indexed strike prices.

In practice, the averaging period δ is between 1 and 6 months, the time delay l from 1 to 3 months and the validity period $q = 1$ month. This last feature comes from the fact that oil forward curves are *monthly*. In the sequel, we will deal with oil-indexed strike prices such that:

Strike price	(δ, l, q) (in months)
(Fo601) and (Go601)	$(6, 0, 1)$
(Fo131) and (Go131)	$(1, 3, 1)$
(Fo311) and (Go311)	$(3, 1, 1)$
(Fo111) and (Go111)	$(1, 1, 1)$

For a valuation with respect to daily spot prices, any classical Markovian solving method becomes computationally unfeasible due to a very high dimension. Our Laguerre approximation-based numerical method provides a 8-dimensional Markovian alternative[3] (using $n = 5$ Laguerre basis functions). We refer the reader to Section 3.3 for the extensions of our pricing algorithm (Lag-LS*) allowing to handle with the above problem.

Remark 4.2.1. As already mentioned, these contracts are most often valued by non Markovian approximate methods very similar to (NM-LS). In addition, some practitioners use a valuation assuming the strike price *deterministic*:

$$\sup_{\tau \leq T} \mathbb{E}\left[\left(S_\tau^g - \bar{X}_\tau^{\text{obs}}\right)^+\right].$$

Namely, $\bar{X}^{\text{obs}}$ is computed from the observed forward curves of the oil prices S^j. In the price model introduced below, this is equivalent to assume a zero volatility coefficient of the price processes implied in the strike price. This gives actually an "intrinsic" value which does not take into account the *stochasticity of the strike price*.
In addition, the following bounds are useful to measure the impact of the *optionality* of the considered contract, in discrete time:

$$\text{Lower bound:} \quad \max_{N_\delta + N_l \leq i \leq N} \mathbb{E}\left[\left(S_{t_i}^{g,\pi} - \bar{X}_{t_i}^{\pi}\right)^+\right]$$

$$\text{Upper bound:} \quad \mathbb{E}\left[\max_{N_\delta + N_l \leq i \leq N} \left(S_{t_i}^{g,\pi} - \bar{X}_{t_i}^{\pi}\right)^+\right]$$

[3] In present 3-dimensional asset framework, the dimension of the state vector is equal to $(n + 3)$.

The commodity spot prices S^g, S^{go} and S^{fo} are assumed to satisfy mean reverting models with one gaussian factor, namely:

$$\forall c \in \{g, go, fo\}, \begin{cases} S_t^c = F^c(0,t) e^{-\frac{\sigma_c^2}{4a_c}\left(1-e^{-2a_c t}\right)+Y_t^c} \\ dY_t^c = -a_c Y_t^c dt + \sigma_c dW_t^c, \ Y_0^c = 0 \end{cases}$$

in which W^g, W^{go}, W^{fo} are three Brownian motions correlated with correlation coefficients denoted by $\rho_{g,go} = d\langle W^g, W^{go}\rangle_t / dt$, $\rho_{g,fo} = d\langle W^g, W^{fo}\rangle_t / dt$ and $\rho_{go,fo} = d\langle W^{go}, W^{fo}\rangle_t / dt$. $(F^c(0,t))_t$, a^c and σ^c are respectively an initial forward curve and constant mean-reverting and volatility coefficients associated to the commodity c.

We use model parameters significant on the European market (calibration on historical prices between October 2007 and October 2008 observed on the Zeebrugge market for the gas price and on the ARA zone[4] for the gas oil and fuel oil prices):

	gas	gas oil	fuel oil
Mean reverting coefficient	$a_g = 50$	$a_{go} = 37$	$a_{fo} = 37$
Volatility	$\sigma_g = 1$	$\sigma_{go} = 0.30$	$\sigma_{fo} = 0.40$

and:

Correlation	gas	gas oil
gas oil	$\rho_{g,go} = 0$	
fuel oil	$\rho_{g,fo} = 0$	$\rho_{go,fo} = 0.8$

The daily gas forward curve $(F^g(0,t))_t$ is represented in Figure 4.14, and we refer in Table 4.5 the gas oil and fuel oil monthly forward curves $(F^{go}(0,t))_t$ and $(F^{fo}(0,t))_t$.

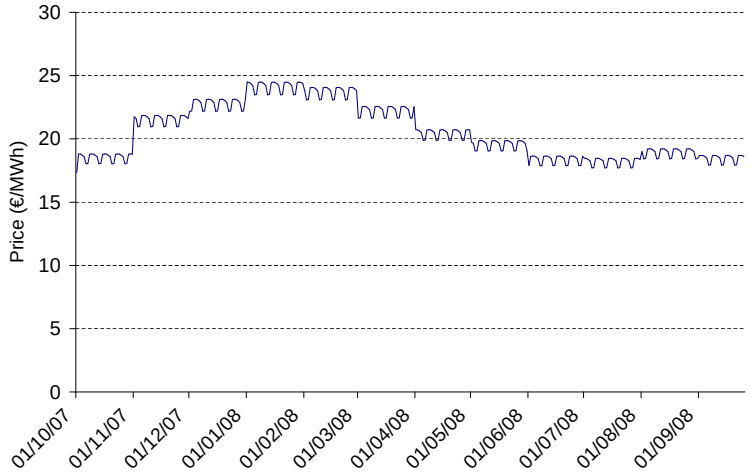

Figure 4.14: Gas daily forward curve.

[4] Amsterdam-Rotterdam-Anvers oil market for West Europe.

	Gas oil (€/Ton)	Fuel oil (€/Ton)
01/10/07	492.16	279.55
01/11/07	491.21	277.90
01/12/07	490.32	276.47
01/01/08	483.72	282.90
01/02/08	482.28	281.49
01/03/08	480.82	280.25
01/04/08	469.46	279.56
01/05/08	467.99	278.31
01/06/08	466.61	277.13
01/07/08	467.60	277.25
01/08/08	466.16	276.30
01/09/08	464.57	274.66
01/10/08	467.44	275.29

Table 4.5: Gas oil and fuel oil monthly forward curves.

4.2.2 Laguerre approximation of indexed strike prices

As an example, we present in Figure 4.15 simulated trajectories of gas, gas oil and fuel oil prices $(S^{g,\pi}, S^{go,\pi}, S^{fo,\pi})$ and corresponding strike price $\bar{X}^{\pi}$ of type (Fo111)-(Go111). Laguerre approximations $M^{n,p_{\text{opt}},\pi}$ of this oil-indexed strike price are reported in Table 4.6.

Updating time	Oil-indexed strike cf. Figure 4.15	Laguerre approximation $n=1$	$n=3$	$n=5$	$n=7$
01/12/07	20.852	21.096	20.815	20.915	20.848
01/01/08	20.954	20.412	20.864	21.052	20.994
01/02/08	19.977	20.390	20.278	20.106	20.020
01/03/08	20.372	20.817	20.370	20.286	20.369
01/04/08	21.132	21.107	20.768	21.096	21.059
01/05/08	20.548	20.257	20.519	20.595	20.646
01/06/08	19.450	19.826	19.605	19.656	19.561
01/07/08	19.427	19.545	19.555	19.362	19.424
01/08/08	19.524	19.279	19.521	19.519	19.514
01/09/08	19.259	19.613	19.423	19.195	19.254
01/10/08	19.789	19.286	19.701	19.705	19.754
Maximal relative error		2.59%	1.72%	1.06%	0.57%

Table 4.6: Simulated trajectory of the oil-indexed strike and its Laguerre approximation.

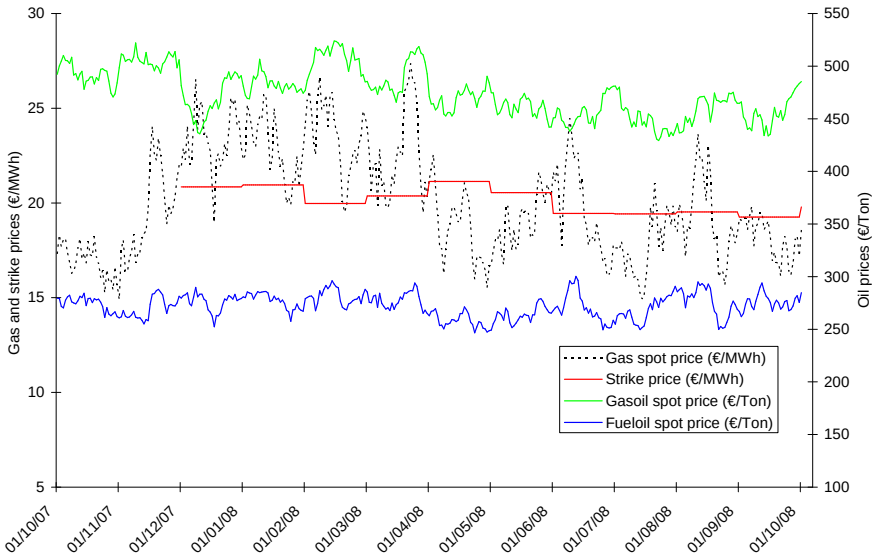

Figure 4.15 Trajectories of gas, gas oil and fuel oil prices and corresponding oil-indexed strike.

Our Laguerre approximation accurately mimics the exact strike price dynamics and we retrieved this accuracy in all the numerical experiments that we performed (with various characteristic numbers (δ, l, q)). We show for example in Figure 4.16 the case of a gas-indexed strike price such that $\delta = 3, l = 1, q = 0$ (in months).

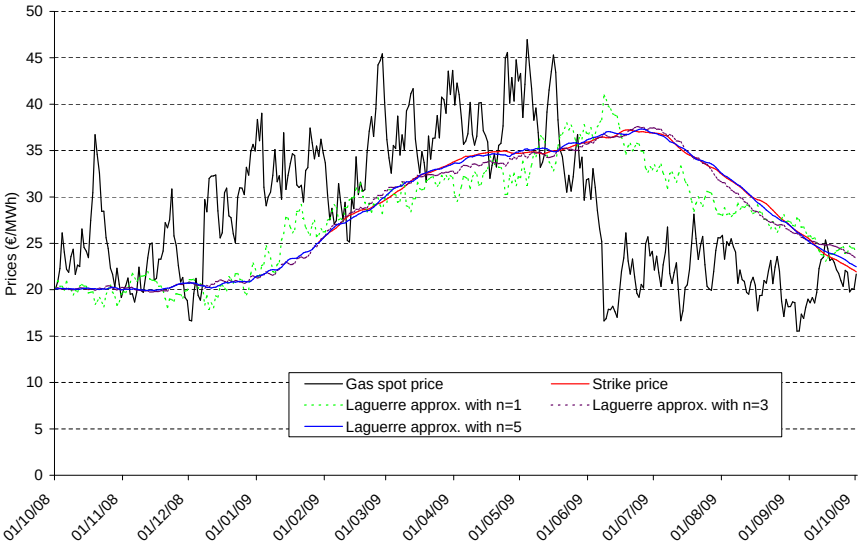

Figure 4.16: Simulated trajectory of the gas price, corresponding strike of type (Gas310) and its Laguerre approximation.

4.2.3 Valuation results

Table 4.7 shows the contract prices computed by (Lag-LS*) and (NM-LS), as well as the other indicative prices introduced in Remark 4.2.1. We used $M = 20$ million of Monte Carlo paths, $b^S = 3$, $b^X = b_k^X = 1$ and $n = 5$ Laguerre functions.

Contract strike type	Bounds	"Intrinsic" value	(NM-LS)	(Lag-LS*)
(601)	[2.183, 4.445]	3.490	3.512	3.513
(131)	[3.847, 7.507]	6.195	6.215	6.226
(311)	[3.912, 7.553]	6.254	6.270	6.277
(111)	[3.976, 8.693]	7.243	7.313	7.321

Table 4.7: Pricing oil-indexed American-style contracts.

On these particular examples, the non Markovian approximate prices are very close (less than 0.2% below) to the prices computed by our method (Lag-LS*). Besides, the stochastic values of such contracts are relatively close to their "intrinsic" values (IV), namely when assuming the strike price deterministic. We present in Table 4.8 some relevant relative differences between the above contract prices.

	Relative differences			
Strike	(NM-LS) w.r.t. (IV)	(Lag-LS*) w.r.t. (IV)	(Lag-LS*) w.r.t. (NM-LS)	Ratio γ
(601)	0.62%	0.66%	0.04%	5.92%
(131)	0.33%	0.50%	0.16%	33.15%
(311)	0.27%	0.37%	0.10%	28.27%
(111)	0.96%	1.07%	0.11%	10.72%

Table 4.8: Pricing oil-indexed American-style contracts.

As we could expect, the stochasticity of strike prices have less impact as the averaging period and/or the time delay increases. Besides, the ratio:

$$\gamma = \frac{|(\text{Lag-LS*}) - (\text{NM-LS})|}{|(\text{Lag-LS*}) - (\text{IV})|}$$

measures the impact of the information loss due to a non Markovian approximation (NM-LS) with respect to the information loss when assuming deterministic strike prices (see also Figure 4.17). Clearly, the impact of path dependence is more important as the averaging window length increases and even more significant in presence of a time delay.

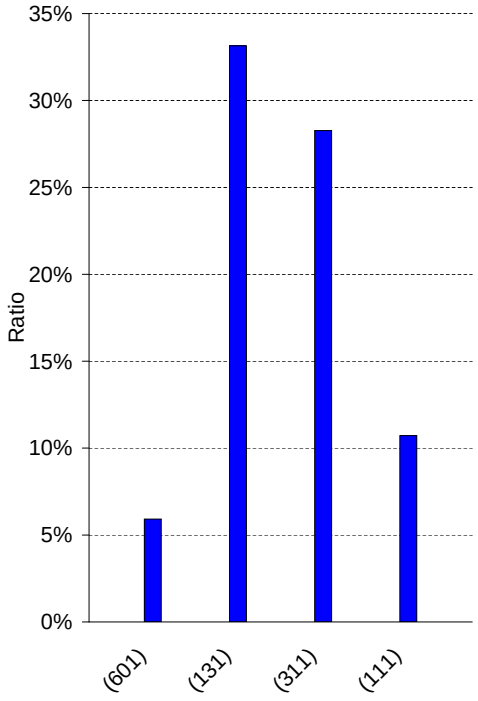

Figure 4.17: Impact of the path dependence w.r.t. the stochasticity of the strike price.

Remark 4.2.2. In present case with one exercise opportunity, the computational time of our method (Lag-LS*) for pricing one year contracts is considerable. We handle with a 8-dimensional state vector and a daily time grid ($\Delta t = 1$ day), and thus, to avoid memory overrun, we need to store in binary files the samples values at each time step of the vector. However, file writing/reading operations lead to a slow computation speed: with the above parameters, the computation of the contract prices takes around 20 hours. In comparison, the computation by (NM-LS) takes around 3 hours.

The generalization of the numerical methods (Lag-LS*) and (NM-LS) to the case of Swing options, with multiple, say $n_{\max}$, exercise opportunities, is straightforward. We just need to combine our Monte Carlo methods (using the Laguerre approximation of the non Markovian one) with an iteration procedure like (2.10) namely, use an inductive scheme on the number of exercise rights left. However, the computational time of such a iteration-based method for pricing Swing options is known to grow *linearly* with with respect to $n_{\max}$, see for example Figure 5.15 in Part II. For pricing realistic gas supplying contracts with a daily exercise opportunity, this would lead to *untractable running times*.

Chapter 5

Perspectives for further research

Convergence of the Laguerre approximation A faster theoretical convergence rate of the Laguerre approximation than a behavior in $\varepsilon(n^{-\frac{3}{4}})$ (cf. Proposition 2.2.2) might probably be obtained, by taking into account the optimal scaling of the Laguerre functions basis. This is a challenging problem since finding an explicit formula to the optimal scale parameter seems difficult. In our particular case of uniformly-weighted moving averages, our numerical observations provide a bound of order $\varepsilon(n^{-1})$, see Figure 3.1.

On the other hand, we have numerically observed that the improved version of our Laguerre approximation-based pricing method (Lag-LS*) shows a monotone convergence in n. It would be interesting to justify this statement from a theoretical viewpoint, which seems however to be a challenging question. Intuitively, this comes from the fact that when increasing n, some information is added and the conditional expectation

$$\mathbb{E}\left[\cdot|\left(S_t^{\pi}, X_t^{p_{\text{opt}},0,\pi}, \ldots, X_t^{p_{\text{opt}},n-1,\pi}\right)\right] \tag{5.1}$$

intuitively tends to the exact value $\mathbb{E}[\cdot|\mathcal{F}_t]$ from below. The monotony on n of the conditional expectations estimators (5.1) is a difficult theoretical question since the martingale structure of the problem disappears (the notion of filtration is lost).

Approximation of moving average options price We find numerically that the non Markovian approximation most often used in practice for pricing standard moving average options constitutes a very good approximation. The theoretical justification of such a behavior remains however an open question.

More generally, a related problem to moving average American options pricing is that of optimal stopping of stochastic differential equations with delay, for which there exist no general numerical solving methods. The Laguerre approximation approach that we introduce is a promising direction for further research.

Valuation of oil-indexed Swing options The numerical method introduced for pricing oil-indexed gas supplying contracts was a first step in the direction of pricing general oil-indexed Swing contracts commonly encountered in the gas market. Its extension to the pricing of such Swing options (which imply a multiple exercise feature) is straightforward, by combining it with an iterative scheme on the number of exercise rights left, see for example (2.10) in Part I. We did not handle with this case for numerical reasons (prohibitive computational time), see Remark 4.2.2.

In the one-exercise case, we have numerically observed that the prices computed by the non Markovian approximate method (NM-LS) were very close to the one computed by our Laguerre approximation-based method. Thus, one could expect that we will *a fortiori* retrieve this same behavior in the multiple-exercise case.

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